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A Strategic User Equilibrium for Independently Distributed Origin-Destination Demands

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Abstract: *This article proposes an extension of the strategic user equilibrium proposed by Waller and colleagues and Dixit and colleagues. The proposed model relaxes the assumption of proportional Origin-Destination (O-D) demand, as it accounts for users' strategic link choice under independently distributed O-D demands. The convexity of the mathematical formulation is proved when each O-D demand is assumed to follow a Poisson distribution independently; link flow distributions and users' strategic link choice are also proved to be unique. Network performance measures are given analytically. A numerical analysis is conducted on the Sioux Falls network. A Monte Carlo method is used to simulate network performance measures, which are then compared to the results computed from the analytical expression. It is illustrated that the model is capable of accounting for demand volatility while maintaining computation efficiency.*

1 INTRODUCTION AND BACKGROUNDS

Realism and computational complexity are both critical factors in transportation planning models and must be equally considered to determine a model's practical applicability. A major complication in transportation modeling is the ability to properly account for the inherent uncertainties such as demand (Axhausen et al., 2002; Richardson, 2003; Bellei et al., 2006; Stopher et al., 2008; Kim et al., 2009), capacity (Brilon et al., 2005; Wu et al., 2010), and connectivity (Iida and Wakabayashi, 1989; Bell and Iida, 1997). Furthermore, as has been noted, uncertainty surrounding these variables directly impacts route choice behavior (Uchida and Iida, 1993; Abdel-Aty et al., 1995; Lam and Small, 2001; Brownstone et al., 2003; de Palma and Picard, 2005). Researchers have focused on the stochastic user equilibrium to account for the uncertainties in a network (Watling, 2002; Nakayama and Takayama, 2003; Meng and Wang, 2008; Bekhor et al., 2009; Sumalee et al., 2011), which requires path enumeration for all Origin-Destination (O-D) pairs. However, little attention has been paid to the user equilibrium that can account for demand volatility while maintaining the computation efficiency. It is, therefore, necessary to incorporate these stochastic elements into models to ensure robust planning capabilities but to do so in a manner that maintains computational tractability. In this section, the review covers the following topics: (1) demand variability and its impact, (2) a brief introduction to the strategic user equilibrium, (3) reliability in transportation, and (4) the maximum entropy method in static user equilibrium.

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Improper consideration of demand variability in planning models can result in gross underestimation of travel time (Waller et al., 2001). Numerous research efforts have focused on the impact of day-to-day stochasticity regarding demand through specific model variations. For example, Clark and Watling (2005) proposed an assignment model with stochastic demand to determine the impact on variance in total system travel time (TSTT). Additionally, Lo and Tung (2003) considered stochastic capacity and proposed a probabilistic user equilibrium. Castillo et al. (2013) further expanded the probabilistic user equilibrium model under capacity uncertainty. Their model can be solved without path enumeration and avoided to use the central limit theorem. The demand volatility can be modeled as various statistical distributions such as Poisson distribution (Bell, 1991; Watling, 2002; Hazelton, 2003), log-normal distribution (Zhou and Chen, 2008; Kuang and Huang, 2013), normal distribution (Shao et al., 2006b), or binomial distribution (Nakayama and Takayama, 2003). Also, there have been many researches to account for demand uncertainty in the network design problem (NDP) perspective (Sumalee et al., 2006; Ukkusuri et al., 2007; Gardner et al., 2008; Sharma et al., 2009; Ukkusuri and Patil, 2009; Yin et al., 2009; Chow and Regan, 2011). The concept of considering the strategic choice in a user equilibrium context was proposed by Nguyen and Pallottino (1989), specifically in the context of transit assignment. This notion was later expanded in the dynamic traffic assignment (DTA) formulation (Hamdouch et al., 2004). Sumalee et al. (2009) proposed a new model for demand and capacity uncertainty where users' strategic choices are obtained via a transformation of cumulative prospect theory. Dixit et al. (2013) proposed a strategic user equilibrium under trip variability, which was further expanded in the linear programming formulation for DTA, by dividing it into strategic stage and realized demand stage (Waller et al., 2013); the link flow will therefore fluctuate corresponding to realized demand.

This article is an extension of the user equilibrium under trip variability proposed by Dixit et al. (2013), the assumption of fixed demand proportion in the original formulation limits the applicability of the model. This article instead proposes a generalization of the strategic user equilibrium; here it is assumed that each O-D demand is independently distributed. The equilibrium assignment problem is to find the link flow distributions that satisfy the user equilibrium criteria (Wardrop, 1952). The equilibrium in this article is a situation where users equilibrate to minimize their expected travel cost based on the demand distribution, and the expected travel cost is less than the cost of any unused paths. The model is named "the strategic user equilibrium," because the model assumes that users make their route choice strategy while knowing the day-to-day demand volatility (e.g., demand is different on each Monday of a week and is also volatile in each day of week), and they tend to be "sticky" to this strategy, that is, their route choice strategy remains fixed regardless of the specific day-to-day realized demand, thus the prediction of link flow will likely vary with the realized demand. As a consequence, disequilibrium may be observed every day, however, the expected link flow will be in the state of equilibrium. Watling and Hazelton (2003) provide an in-depth discussion on the definition of equilibrium, and also identify the existence of disequilibrium due to the underlying variations such as demand. This model therefore captures the demand uncertainty and its impact on the travel time reliability, while maintaining the computation tractability and simplicity within the static user equilibrium.

It is due to these uncertainties in the network that has led to the concerns with ensuring reliability. In part, this has come about due to the finding that road users tend to value reliability at about the same magnitude as delays (Abdel-Aty et al., 1995; Bates et al., 2001; Brownstone et al., 2003; Asensio and Matas, 2008; Dixit et al., 2013; Uchida, 2015), hence the impact of travel time variation should not be neglected. The reliability of travel time may be affected by capacity degradation, for example, Lo et al. (2006) extended the user equilibrium to a reliability-based user equilibrium (RUE) to account for travel time reliability by adding a safety margin in the travel time budget. The RUE was further expanded to incorporate multimodal transport (Shao et al., 2006b; Fu et al., 2014), travelers' perception errors (Clark and Watling, 2005; Shao et al., 2006b), and network uncertainties (Shao et al., 2006b; Zhou and Chen, 2008). Furthermore, extensive previous research has considered travel time reliability as a result of path flow correlation (Clark and Watling, 2005; Lam et al., 2008; Shao et al., 2013), where *a priori* information on path flows and path enumeration is required, which may be computationally complicated. Under the framework, the correlation between links is considered hence path travel time is nonadditive (Fan et al., 2005; Dong and Mahmassani, 2009). Path travel time can be derived from link travel time covariance matrix (Sen et al., 2001; Xing and Zhou, 2011; Shahabi et al., 2013), temporal and spatial correlations (Miller-Hooks and Mahmassani, 2003; Gao and Chabini, 2006), or simulation-based approaches (Ji et al., 2011; Huang and Gao, 2012; Zockaie et al., 2013; Zockaie et al., 2014). Some research also focuses on the penalties due to late arrival and the corresponding route choice (Watling, 2006; Chen and Zhou, 2010). Generally, the stochastic variations in transportation systems can be attributed to variations in demand, capacity, or driving behavior (Shao et al., 2006a; Siu and Lo, 2008; Van Lint et al., 2008), which variation is the dominant source of travel time variation is still an open question. This article aims to account for the demand uncertainty and travel time variation caused by it. The risk of variation of travel demand and its effects on route choice are explained by Uchida and Iida (1993), who developed a new assignment model to consider the impact of variation in travel time. It should also be noted that the impact of not considering these uncertainties leads to significant biases and errors (Duthie et al., 2011). Hence, it is important to incorporate the critical realities of demand uncertainties in our transportation models for travel behavior, so that consistent decisions can be made based on cost-benefit analysis associated to improving reliability by affecting variations in demand. In this article, we account for the link travel time reliability as a variable caused by demand uncertainty and formulate the strategic user equilibrium as a convex optimization problem, which can be efficiently solved by some numerical methods such as the widely used Frank-Wolfe (F-W) algorithm.

The link flows are uniquely defined under the static user equilibrium framework originally formulated by Beckmann's transformation (Beckmann et al., 1956), however, multiple path flow solutions are possible depending on the model assumptions and methodologies (Larsson et al., 2001). Analyses based on an arbitrary choice among the infinite number of possible route flow solutions could cause inconsistencies in transportation planning models such as O-D matrix estimation, emission analysis, and many more. Rossi first suggested that the entropy-maximizing pattern is the most likely route flow pattern; the implication is to split flows evenly across all user equilibrium (UE) paths (Rossi et al., 1989). Later, the stability and continuity of this approach are theoretically proved (Lu and Nie, 2010). However, the scalability of this approach remains an issue due to that the set of UE paths needs to be enumerated. Janson (1993) provided a link-based problem which is equivalent to the maximum entropy approach, and proposed a modified F-W algorithm for the implementation of the model. However, the mathematical proof of its equivalency to the original maximum entropy approach is pointed out to be inconsistent by Akamatsu (1997). The efficiency of the maximum entropy approach was later improved in various ways: Bar-Gera applied the condition of proportionality (Bar-Gera and Boyce, 1999; Bar-Gera, 2010) and found an approximate set of UE paths (Bar-Gera, 2006). Some dual methods are exploited to find the most likely path flows (Bell and Iida, 1997; Larsson et al., 2001), in addition, Kumar and Peeta (2015) proposed an entropy weighted average method to minimize the expected Euclidean distance from all other path flow solution vectors of the static user equilibrium. However, little attention has been paid to these two aspects: first, most of the methods rely on Stirling's approximation to convert the entropy into a continuous function, and is subjected to the limitations of this approximation (Schrödinger, 1957); second, the path flow is treated as a deterministic variable, however, it is possible to define the entropy of a probability distribution in information theory and to interpret this as a measure of uncertainty associated with that distribution. The maximum entropy of probability distribution method in this model has addressed both issues by considering the entropy of path flow distributions, and can eventually provide users' O-D specific link choices, which will be referred to as the strategic link choice in this article.

The article is organized as follows. Section 2 presents the mathematical model. The link flow distribution is proved to be unique under this framework, which is extremely useful for transport planning and policies. Users' strategic link choice is also proved to be unique under the further assumption of maximum entropy, which provides the possibility of applying this user equilibrium in many other transportation models such as O-D trip matrix estimation, environmental impact analysis and fuel consumption analysis. The performance measures of a network are analytically derived with respect to the expected link flow; consequently, the impact of demand variability can be computed directly from the mathematical expression. An implementation algorithm is also demonstrated. In Section 3, the numerical analysis demonstrates that these analytically derived performance measures are closely approximated to the simulated results, so the model captures the demand volatility while maintaining the simplicity and tractability of traditional deterministic traffic equilibrium methods. The importance of incorporating uncertainty in maximum entropy method is presented. Some discussions of model limitations and possible future research are presented in Section 4.

The highlights of this article include:

(1) It accounts for demand uncertainty in users' routing mechanism, and its direct impact on link flow and link travel time variation.
(2) It introduces the maximum entropy of a random variable in the assignment model, instead of a deterministic variable in the majority of the literature.
(3) Network performance measures are provided analytically, which enhances computation efficiency.
(4) The uniqueness of link flow distribution and strategic link choice are guaranteed, which ensures its applicability in transportation planning process.

Table 1
Summary of notations

N	Link (index) set.
M	O-D pair (index) set.
K	Path set.
l_n	Flow variable for link n.
$t_n()$	The function of travel time on link n.
$\overline{t_n}$	The expected travel time on link n.
t_{nf}	The free flow travel time on link n.
C_n	The capacity on link n.
p_k^m	Users' path choice on path k, connecting O-D pair m.
d_n^m	Users' strategic link choice, which represents the proportion of O-D pair demand T_m on link n.
h_k^m	The flow on path k, connecting O-D pair m.
M_n^θ	The θ^{th} raw moment of the link flow distribution on link n.
$E()$	The expectation of a variable.
$var()$	The variance of a variable.
$G(\,)$	The probability density function of a variable.
q_m	Proportion of total trips that are between O-D pair m; $1 = \sum_{\forall m} q_m$.
T^m	Demand variable for O-D pair m.
λ_n	The parameter of the Poisson distribution for flow on link $n[\lambda] = [\lambda_1, \ldots, \lambda_n]^T$.
s_m	The parameter of the Poisson distribution for O-D pair demand T^m.
$g(l_n; \lambda_n)$	The probability mass function of a Poisson distributed variable l_n, defined by the parameter λ_n.
$\delta_{n,k}^m$	Link-path indicator variable. $\delta_{n,k}^m = \begin{cases} 1 & if\ link\ n\ is\ on\ path\ k\ between\ OD\ pair\ m \\ 0 & otherwise \end{cases}$
K_m	The path set for O-D pair m.
K_m^{UE}	The shortest path set for O-D pair m.
$S()$	The entropy of a variable.
α	The parameter of the BPR function.
$\nabla z()$	The gradient of an objective function $z()$.

(5) Users are assumed to stick to their route choice regardless of day-to-day demand volatility, which mimics the "sticky" behavior of travelers, especially commuters.
(6) A solution algorithm is proposed to compute the link flow parameter and the strategic link choice, without requiring path enumeration.

2 PROBLEM FORMULATION

This section defines the mathematical concept of the strategic user equilibrium model (referred to as StrUE in this article). Table 1 shows a summary of notations used in the article.

2.1 Model formulation

This model is an extension of the strategic user equilibrium proposed by Dixit et al. (2013) and Waller et al. (2013), which was formulated as

$$\min z([x]) = \sum_{n \in N} \int_0^{\infty} \int_0^{x_n} t_n(yT)\, G(T)\, dy dT \qquad (1)$$

subject to

$$\sum_k p_k^m = 1, \qquad \forall m \in M \qquad (2)$$

$$p_k^m \geq 0, \qquad \forall m \in M, k \in K_m \qquad (3)$$

$$x_n = \sum_m \sum_k p_k^m \delta_{n,k}^m q_m\ ,\ \forall n \in N \qquad (4)$$

In this formulation, users' strategic choice is denoted as the link proportions x_n, that is, the link flow divided by the total demand T. The $[x]$ represents a vector of all the link proportions and y is a dummy variable for integration. Each O-D demand is the fixed demand proportion q_m multiplied by the total demand T, that is, each O-D demand is proportional to the total demand, and the proportions are constants. This implies that all O-D demands are perfectly correlated, that is, each O-D demand always inflates or deflates by the same percentage. The total demand is assumed to follow a certain distribution. The objective function is the sum of the integrals of the expected value of the link cost functions; the link proportions are proved to be unique under the assumption of fixed demand proportions. Note that the link proportion does not change in line with the realized total demand day-to-day, while link flow will vary every day corresponding to the realized total demand. Further explanation and details can be found in Dixit et al. (2013).

However, the assumption of fixed demand proportion limits the applicability of the model, therefore, this article proposes a generalization of the strategic user equilibrium, and here it is assumed that each O-D demand is independently distributed. The equilibrium assignment problem is to find the link flow distributions that satisfy the user equilibrium (Wardrop, 1952) criterion. Due to the introduction of demand volatility and expected travel time, in this article, it is defined as:

Definition 1. *The strategic user equilibrium is defined such that the expected travel costs are equal on all used paths, and this commonly expected travel time is less than the actual expected travel time on any unused path. In other words, given user equilibrium expected path cost, any deviation from the existing expected path flows cannot reduce the expected path cost.*

Based on Definition 1, the equilibrium condition can be formulated by the following link-based mathematical program:

$$\min z([l]) = \sum_{n \in N} \int_0^{l_n} \int_0^{+\infty} t_n(w)\, G(l_n)\, dl_n dw \quad (5)$$

subject to

$$\sum_{k \in K_m} p_k^m = 1 \qquad \forall m \in M \quad (6)$$

$$p_k^m \geq 0 \qquad \forall m \in M, k \in K_m \quad (7)$$

$$l_n = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m T^m \qquad \forall n \in N \quad (8)$$

where w is a dummy integration variable which represents link flow. In this formulation, the objective function is the sum of the integrals of the expected value of the link cost functions from 0 to expected link flow; its equivalence to the variational inequality for user equilibrium will be proved later. The behavioral implication and its equivalence of this formulation to user equilibrium will be shown in this section. Equation (6) represents a set of flow conservation constraints, that is, the sum of path flow proportions for every O-D pair m should be equal to 1, which preserves the trips out of and in each O-D centroid. Equation (7) indicates that the path flow must be non-negative. Equation (8) indicates the link flow in terms of path flows and O-D demand. The topology of link, path and O-D pairs of the network is represented by Equation (8).

As $\delta_{n,k}^m$ is dependent only on the network topology and is a constant, $\sum_{k \in K_m} p_k^m \delta_{n,k}^m$ can be integrated into one term d_n^m which is defined here as the users' strategic link choice. It represents the proportion of O-D pair demand T_m traversing link n, the strategic link choice indicates users' link choice disaggregated by O-D pairs, which is extremely important in many transportation models such as O-D matrix estimation, emission analysis, and NDP. As aforementioned in this article, the users will stick to this strategic link choice once it is made. The strategic link choice will be discussed further later in this article.

$$d_n^m = \sum_{k \in K_m} p_k^m \delta_{n,k}^m \qquad \forall m \in M,\ \forall n \in N \quad (9)$$

Under the proposed framework, two further statistical assumptions are made here.

A1. The actual O-D demand varies day-to-day and follows a Poisson distribution independent of each other (Bell, 1991; Hazelton, 2003; Clark and Watling, 2005; Appiah, 2009; Bera and Rao, 2011) defined by the probability mass function: $T^m \sim g(T^m; s_m)$, where $s_m > 0$. This assumption will be discussed later in the article.

A2. Conditional on the realized demand on any given day, each user (driver) is assumed to choose independently between the alternative paths with a fixed strategic path choice p_k^m.

Before proceeding to the mathematical proof, some clarifications are made here.

Note that the uniqueness and equivalence conditions are guaranteed by the assumption of Poisson distributed demands, and the non-negativity of Poisson distribution is consistent with real-world positive demands. The actual demand distribution may vary depending on network type, time frame, and many other factors; which distribution best fits the actual demand is still an open question. In some cases when demand is considered to not follow the Poisson distribution, other distributions are also applicable as long as they are non-negative and preserve the uniqueness and equivalence conditions. If the depicted distribution is not supported on the positive semi-infinite interval, the truncated distribution techniques may be applied, such as the truncated normal distribution, which is defined on the domain of positive real numbers.

Each O-D demand follows a Poisson distribution; however, each realized demand T_m^* is a constant, that is, on that day, there are T_m^* travelers between O-D pair m. Each of the T_m^* travelers will choose between the k alternative paths, each with a probability p_k^m, note that the strategic path choice should be treated as a probability instead of a constant. One possible misunderstanding is to treat the strategic path choice as a constant; in this case, the path flow variable would become $h_k^m = p_k^m T^m$, namely, a constant multiplied by a Poisson variable. As $0 \leq p_k^m \leq 1$, the path flow would not follow a Poisson distribution, which is inconsistent with our assumption.

Proposition 1. *Given assumptions 1 and 2, the unconditional path flow h_k^m follows a Poisson distribution $g(h_k^m; p_k^m s_m)$ independently. The unconditional link flow l_n also follows a Poisson distribution.*

Proof. Assumptions 1 and 2 together imply that for each $m \in M$, conditional on a realized demand T_m^*, each user has a probability of choosing path k. By definition, the probability of observing a set of path flow $[h_1^{m*}, h_2^{m*}, \ldots, h_k^{m*} | m \in M]$ has a multinomial distribution

$$P\left(h_1^m = h_1^{m*}, h_2^m = h_2^{m*} \ldots h_k^m = h_k^{m*} \,\middle|\, T_m^*\right) = \frac{T_m^*!}{h_1^{m*}! h_2^{m*}! \ldots h_k^{m*}!} (p_1^m)^{h_1^{m*}} (p_2^m)^{h_2^{m*}} (p_k^m)^{h_k^{m*}} \quad (10)$$

$$\forall m \in M, k \in K_m$$

where $\sum_1^k h_k^{m*} = T_m^*$. In StrUE, we are more interested in the unconditional flows; here the path flow conditional on $T_m = T_m^*$ has a multinomial distribution, and T_m follows a Poisson distribution, then from the relationship of unconditional and conditional probability we obtain

$$P\{(\, h_1^m = h_1^{m*}\,,\; h_2^m = h_2^{m\,*} \ldots\; h_k^m = h_k^{m*}\,) \cap \left(\sum_1^k h_k^{m*} = T_m^* \right) = P\,(\, h_1^m = h_1^{m*}\,,\; h_2^m = h_2^{m*} \times \ldots\; h_k^m = h_k^{m*} \;\left| T_m^* \right. = \sum_1^k h_k^{m*} \right) * P \left(T_m^* = \sum_1^k h_k^{m*} \right) = \frac{\left(\sum_1^k h_k^{m*}\right)!}{h_1^{m*}!h_2^{m*}! \ldots h_k^{m*}!} \times (p_1^m)^{h_1^{m*}} (p_2^m)^{h_2^{m*}} (p_k^m)^{h_k^{m*}} \frac{e^{-s_m} s_m^{\sum_1^k h_k^{m*}}}{\left(\sum_1^k h_k^{m*}\right)!}, \quad \forall m \in M, k \in K_m \tag{11}$$

Because $\sum_{k \in K_m} p_k^m = 1$, the above equation can be rearranged as

$$\frac{\left(\sum_1^k h_k^{m*}\right)!}{h_1^{m*}!h_2^{m*}! \ldots h_k^{m*}!} (p_1^m)^{h_1^{m*}} (p_2^m)^{h_2^{m*}} (p_k^m)^{h_k^{m*}} * \frac{e^{-s_m} s_m^{\sum_1^k h_k^{m*}}}{\left(\sum_1^k h_k^{m*}\right)!} = \frac{e^{-p_1^m s_m} \left(p_1^m s_m\right)^{h_1^{m*}}}{h_1^{m*}!} * \frac{e^{-p_2^m s_m} \left(p_2^m s_m\right)^{h_2^{m*}}}{h_2^{m*}!} \ldots \frac{e^{-p_k^m s_m} \left(p_k^m s_m\right)^{h_k^{m*}}}{h_k^{m*}!} = \prod_1^k g\,(h_k^m; p_k^m s_m) \qquad \forall m \in M, k \in K_m \tag{12}$$

Q.E.D.

Note that the proof of Proposition 1 is also known as "thinning of a Poisson process," a general and more detailed proof can be found in Corollary 9.17 in Boucherie and Serfozo (2001) or in Clark and Watling (2005) and Castillo et al. (2014). The right-hand side of Equation (12) is a product of a series of Poisson variables h_k^m, whose parameters are $p_k^m s_m$. So we have proved that each unconditional path flow follows an independent Poisson distribution; and note that the conclusion of independent path flow does not violate the flow conservation constraints as the path flow conditional on a realized demand still must sum up to each realized demand. From Equation (8), we have

$$l_n = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m T^m = \sum_{m \in M} \sum_{k \in K_m} h_k^m \delta_{n,k}^m\,, \quad \forall n \in N \tag{13}$$

The link-path indicator variable can only be either 0 or 1; therefore, link flow is the sum of several independent Poisson distributions, which also follows a Poisson distribution (Lehmann and Romano, 2006). The parameter of link flow distribution is then defined by the equations below

$$l_n \sim g\,(\lambda_n) \tag{14}$$

$$\lambda_n = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m s_m = \sum_{m \in M} \sum_{k \in K_m} E\,(h_k^m)\, \delta_{n,k}^m \quad \forall n \in N \tag{15}$$

Corollary 1. *The flow conservation constraint also holds for expected path flow and variance of path flow.*

Proof. Each O-D demand follows a Poisson distribution with parameter $s_m > 0$, which is the expected O-D demand. Multiply both sides of the flow conservation constraints by the expected demand for O-D pair m, and as the summation is taken over k instead of m, Equation (6) can be rewritten as

$$\sum_{k \in K_m} p_k^m = 1 \rightarrow \sum_{k \in K_m} p_k^m * s_m = 1 * s_m = \sum_{k \in K_m} E\,(h_k^m) \quad , \forall m \in M \tag{16}$$

Also, if we look at the variance of path flows, according to Proposition 1 and based on the basic property of variance, we have

$$Var\;(T_m) = Var \left(\sum_{k \in K_m} h_k^m \right) = \sum_{k \in K_m} Var(\,h_k^m), \forall m \in M \tag{17}$$

Proposition 2. *The parameters of the link flow distributions are unique under A1 and A2. That is, there exists a unique solution to the mathematical program defined in Equations* (5) *to* (8).

Use BPR (The Bureau of Public Roads) function as the travel cost function

$$t_n\;(l_n) = t_{nf} \left(1 + \alpha \left(\frac{l_n}{C_n} \right)^4 \right) \qquad \forall n \in N \tag{18}$$

Note that the travel cost is positive, and due to the equilibrium conditions that all used paths have minimum travel costs, a path with cycles will not be considered as the user equilibrium paths. Because the link flow variable l_n follows the discrete Poisson distribution with a probability mass function $g(l_n; \lambda_n)$ which is defined on positive integers, based on the aforementioned proposition and corollary, the objective function of StrUE in Equations (5) to (8) can then be rewritten as

$$\min z([\lambda]) = \sum_{n \in N} \int_0^{\lambda_n} \sum_{l_n=0}^{\infty} \left[t_{nf} \left(1 + \alpha \left(\frac{l_n}{C_n} \right)^4 \right) \right] g(l_n; \lambda_n)\, d\lambda_n = \sum_{n \in N} \int_0^{\lambda_n} \left(t_{nf} + \alpha t_{nf} \left(\frac{1}{C_n} \right)^4 M_n^4 \right) d\lambda_n \tag{19}$$

subject to

$$\sum_{k \in K_m} E(h_k^m) = s_m \qquad \forall m \in M \tag{20}$$

$$E(h_k^m) \geq 0 \qquad \forall m \in M, k \in K_m \tag{21}$$

$$\lambda_n = \sum_{m \in M} \sum_{k \in K_m} E(h_k^m) \delta_{n,k}^m \quad \forall n \in N \tag{22}$$

where M_n^4 is the fourth raw moment of the corresponding Poisson distribution for link n

$$M_n^4 = \lambda_n \left(1 + 7\lambda_n + 6\lambda_n^2 + \lambda_n^3 \right), \forall n \in N \tag{23}$$

The gradient of the objective function is

$$\nabla z([\lambda]) = \begin{bmatrix} \dfrac{\partial z([\lambda])}{\partial \lambda_1} \\ \vdots \\ \dfrac{\partial z([\lambda])}{\partial \lambda_n} \end{bmatrix} = \begin{bmatrix} \sum_{l_1=0}^{\infty} \left[t_{1f} \left(1 + \alpha \left(\frac{l_1}{C_1} \right)^4 \right) \right] g(l_1; \lambda_1)\, d\lambda_1 \\ \vdots \\ \sum_{l_n=0}^{\infty} \left[t_{nf} \left(1 + \alpha \left(\frac{l_n}{C_n} \right)^4 \right) \right] g(l_n; \lambda_n)\, d\lambda_n \end{bmatrix} = \begin{bmatrix} \left(t_{1f} + \alpha t_{1f} \left(\frac{1}{C_1} \right)^4 M_1^4 \right) \\ \vdots \\ \left(t_{nf} + \alpha t_{nf} \left(\frac{1}{C_n} \right)^4 M_n^4 \right) \end{bmatrix} = \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} \tag{24}$$

where $\overline{t_n}$ is the expected travel time on link n. Because the expected travel time depends only on the link flow distribution

$$\frac{\partial z([\lambda])}{\partial \lambda_a \lambda_b} = \begin{cases} \dfrac{\partial \left(t_{nf} + \alpha t_{nf} \left(\frac{1}{C_n} \right)^4 M_b^4 \right)}{\partial \lambda_a}, & if\ a = b \\ 0 & if\ a \neq b \end{cases} \quad \forall a,\ b \in N \tag{25}$$

The Hessian matrix of the objective function can be expressed as:

$$Hessian = \begin{bmatrix} 1 + 14\lambda_1 + 18\lambda_1^2 + 4\lambda_1^3 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 + 14\lambda_a + 18\lambda_a^2 + 4\lambda_a^3 \end{bmatrix} \quad a \in N \tag{26}$$

As $\lambda_a > 0$, the Hessian matrix is positive definite, in addition, all constraints are linear. Therefore, the objective function is strictly convex and has a unique minimum with respect to link flow distribution. The uniqueness is extremely important to ensure stability of the project rankings in planning models. Note that as the θ^{th} raw moment of a Poisson distribution can always be expressed in a polynomial form, other travel cost functions may also be used as long as they are monotonically increasing with respect to link flow. For example, changing the exponent parameter for BPR function to two or there clearly does not change the positive-definiteness of the Hessian matrix of the objective function.

Q.E.D.

2.2 Proof of equivalency

In this part, we will demonstrate that the mathematical formulation is equivalent to the notion of strategic user equilibrium as proposed by Definition 1.

Theorem 1. *The mathematical program defined in Equations* (19) *to* (22) *is equivalent to the strategic user equilibrium condition defined in Definition 1.*

Proof. For convenience, here we number the paths distinctly and consecutively by dropping the subscripts k and m and replacing them with e, that is, there are $e = \sum_{m \in M} \sum_{k \in K_m} k$ (i.e., the sum of all possible paths across all O-D pairs) distinct paths for the network. Let H denote the path flow vector and $\bar{H}$ represent the corresponding expected flow vector. Equation (22) can then be written in the following matrix form:

$$H = \begin{bmatrix} h_1 \\ \vdots \\ h_e \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} E(h_1) \\ \vdots \\ E(h_e) \end{bmatrix},$$
$$A = \begin{bmatrix} a_{11} & \dots & a_{1e} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{ne} \end{bmatrix} \rightarrow [\lambda] = A * \bar{H} \tag{27}$$

where $[\lambda]$ is a vector of $\lambda_n (n \in N)$, let $[\lambda^*]$ be the vector of a local minimum, A is the link-path incidence matrix which represents whether link n is included in path e or not, the entry of A is either 0 or 1. Equation (27) indicates the constraints with respect to λ_n; clearly, the constraints are convex with respect to λ_n. The convexity of the constraints implies that another vector $[\widetilde{\lambda}] = q[\lambda] + (1-q)[\lambda^*]$ is also a feasible set for $0 \le q \le 1$. Thus,

$$\begin{aligned} [\widetilde{\lambda}] - [\lambda^*] &= q[\lambda] + (1-q)[\lambda^*] - [\lambda^*] \\ &= q([\lambda] - [\lambda^*]) \end{aligned} \tag{28}$$

This indicates a step move in a feasible direction. For a q that is small enough, the convexity of the objective function tells us that

$$q([\lambda] - [\lambda^*]) \nabla z([\lambda^*]) \ge 0 \tag{29}$$

where $\nabla z([\lambda^*])$ is the gradient of the objective function. Dividing both sides of Equation (29) by q yields

$$([\lambda] - [\lambda^*]) \nabla z([\lambda^*]) \ge 0 \tag{30}$$

As shown above in the calculation in Equation (24), the gradient of the objective function is a vector of the expected travel cost on link n, and the expected cost of any path e consists of the sum of the expected costs of the constituent links

$$O^* = \begin{bmatrix} o_1 \\ \vdots \\ o_e \end{bmatrix} = A^T * \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} \tag{31}$$

where O^* represents the expected path cost vector. Therefore, from Equations (24), (30), and (31), we have shown that

$$\begin{aligned} \left(\bar{H} - \bar{H}^*\right)^T O^* &= \left(\bar{H} - \bar{H}^*\right)^T * A^T * \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} \\ &= ([\lambda] - [\lambda^*])^T * \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} \ge 0 \end{aligned} \tag{32}$$

Therefore, the formulation is equivalent to: given user equilibrium expected path cost, any deviation from the existing expected path flows cannot reduce the expected path cost. Another way to say that the StrUE is reached when the expected travel times are equal on all used paths, and this common expected travel time is less than the actual expected travel time on any unused path.

The uniqueness of link flow is extremely important to ensure the model's applicability in transportation planning process. Note that in the proposed framework, other distributions of O-D demand may also be assumed, provided that the gradient of the equivalent optimization objective function represents the expected travel cost on each path, which is necessary to guarantee the variational inequality of equilibrium. In addition, the Jacobian of the link cost functions with respect to link flow must be positive definite to assure the uniqueness of expected link flow. That is, the cost function should be monotonically increasing in terms of link flow, and the dominant effect on the cost of a link should be the flow. The convexity of constraints in the mathematical program is sufficient and necessary for the existence of equilibrium. A further discussion of the variational inequality and the equivalence conditions can be found in Cascetta (2009), which also explains another way to model variable demand.

2.3 Analytical expression

One of the strengths of this model is that we have tied back the demand uncertainty to the mathematical expression, which can substantially decrease computation steps needed. Note that the link flow parameter λ_n and users' strategic link choice are derived from the numerical method and are treated invariant. They are substituted into those mathematical expressions to calculate those performance measures.

The expected travel time on a link is given by

$$\overline{t_n} = E[t_n(\lambda_n)]$$

$$= \sum_{l_n=0}^{\infty} \left[t_{nf} \left(1 + \alpha \left(\frac{l_n}{C_n} \right)^4 \right) \right] g\left(l_n; \lambda_n\right) d\lambda_n$$

$$= t_{nf} \;+ \alpha t_{nf} \left(\frac{1}{C_n} \right)^4 M_n^4 \quad, \forall n \in N \tag{33}$$

The variance of travel time on a link is given by

$$\text{var}\left[t_n\left(\lambda_n\right)\right] = E\left[t_n^2\left(\lambda_n\right)\right] - E\left[t_n\left(\lambda_n\right)\right]^2$$

$$= \alpha^2 \, t_{nf}^2 \left(\frac{1}{C} \right)^8 \left[M_n^8 - \left(M_n^4\right)^2 \right], \forall n \in N \tag{34}$$

$$M_n^\theta = \sum_{i=1}^{\theta} \lambda_n^i \begin{Bmatrix} \theta \\ i \end{Bmatrix}, \forall n \in N \tag{35}$$

where the braces in Equation (35) denote the Stirling's number of the second kind. Therefore, the raw moment of link flow is monotonically increasing with respect to λ_n, therefore, it can be characterized that the variance of travel time increases as the expected travel time increases, which is a commonly observed phenomenon on transportation networks around the world (Van Lint et al., 2008; National Research Council, 2013).

Once we have λ_n, the StrUE model can also estimate the $TSTT$ analytically, which are shown in the equations below. Some characteristics of these expressions will be discussed in Section 3.

$$E\ (TSTT) = \sum_{n \in N} \sum_{l_n=0}^{\infty} \left[l_n t_{nf} \left(1 + \alpha \left(\frac{l_n}{C_n} \right)^4 \right) \right] g\left(l_n; \lambda_n\right) d\lambda_n = \sum_{n \in N} t_{nf} M_n^1 + \alpha t_{nf} \left(\frac{1}{C_n} \right)^4 M_n^5 \tag{36}$$

$$var\ (TSTT) = E[(TSTT - E\left[TSTT\right])^2] = E\left(TSTT^2\right) - E^2\left(TSTT\right) = \sum_{n \in N} t_{nf}^2 \left[M_n^2 - \left(M_n^1\right)^2 \right] + \alpha^2 t_{nf}^2 \left(\frac{1}{C_n} \right)^8 \left[M_n^{10} - \left(M_n^5\right)^2 \right] + 2\alpha t_{nf}^2 \left(\frac{1}{C_n} \right)^4 \left[M_n^6 - M_n^1 M_n^5 \right] \tag{37}$$

2.4 The most likely strategic link choice

Under the strategic user equilibrium, the link flows are uniquely defined, however, the strategic link choice matrix (sometimes referred to as the assignment map or O-D-specific link choice in other papers) is not unique. That is, there might be multiple sets of strategic link choice that can produce the same link flows. Although if we run the F-W algorithm and store all the temporary shortest paths for each iteration, the F-W algorithm may provide different link choice each time, while producing the same link flows. The strategic link choice is extremely useful especially in transportation planning models such as O-D estimation problem, emission analysis, and so forth. In these cases, only having the aggregated link flows is not sufficient, hence a uniquely determined strategic link choice is required to ensure the stability and applicability of this model. To address this issue, some researchers have proposed the following maximum entropy optimization problem to determine the most likely path flows (Rossi et al., 1989; Janson, 1993; Larsson et al., 2001) (which can provide strategic link choice subsequently), where entropy is defined as the number of possible route choice decisions made by individual travelers, and path flow and O-D demand are treated as deterministic variables, normally only the expected path flow and O-D demand are considered

$$S_{system} = \prod_{m \in M} \frac{E\left(T_m\right)!}{\prod_{k \in K_m} E\left(h_k^m\right)!} \tag{38}$$

In principle, entropy gives the number of possible route choice decisions made by individual travelers within a specific route flow solution. The objective is to maximize the entropy of the system (the sum of the entropy of all shortest paths), which can be formulated as the following equivalent mathematical program:

$$Max: \; -\sum_{m \in M} \sum_{k \in K_m^{UE}} E\left(h_k^m\right) \ln\left[E\left(h_k^m\right)\right] \tag{39}$$

subject to

$$\sum_{k \in K_m^{UE}} E\left(h_k^m\right) = E\left(T_m\right) \qquad \forall m \in M \tag{40}$$

$$h_k^m \geq 0 \qquad \forall m \in M, \; k \in K_m^{UE} \tag{41}$$

$$l_n = \sum_{m \in M} \sum_{k \in K_m^{UE}} p_k^m E\left(h_k^m\right) \qquad \forall\, n \in N \tag{42}$$

Note that only the shortest paths for the strategic user equilibrium are considered; the problem is based on that the user equilibrium is solved and the equilibrium link flow is obtained. Solving the Lagrangian program above shows that the most likely path flow follows the logit assignment, where the "link cost" is the Lagrangian multipliers corresponding to Constraint 41 (Akamatsu, 1996; Akamatsu, 1997). However, the above formulation may suffer from two limitations: First, the transformation to the equivalent mathematical program relies on Stirling's approximation to convert the entropy into a continuous function, and is subjected to the limitations of this approximation (Schrödinger, 1957). Second, the path flow is treated as a deterministic variable hence it neglects the volatility in O-D demand and path flows. It is possible to define the entropy of a probability distribution in information theory and to interpret this as a measure of uncertainty associated with that distribution. In the formulation below, we will show that if path flow and demand volatility are considered, the path flow will *not* follow the logit assignment, therefore, Dial's algorithm may not be applicable under this consideration.

Definition 2. *The entropy of any path flow state (Hereby referred to as path flow distribution) is the measure of randomness or uncertainty of locating an individual random network user. All possible states of this distribution are considered. The higher the number of possible states, the higher is the randomness or uncertainty of locating an individual network user in that path flow state.*

Based on the definition, the following assumption is made.

A.3. Under the strategic user equilibrium conditions, users make their strategic path choice (and the corresponding strategic link choice), to maximize the probability distribution entropy of the system.

In the aforementioned formulation, the strategic path choice is not uniquely defined, that is, there may be several sets of path flows that can provide the same link flow distributions. However, the capability of estimating path flow uniquely may be important in various transportation models. Hence, extending the notion of entropy maximization method used in many previous researches (Rossi et al., 1989; Akamatsu, 1997; Kumar and Peeta, 2015), here the entropy of a random variable (instead of a deterministic value) which follows a certain statistical probability distribution is defined as (Ochs, 1976)

$$S\,(h_k^m) = -\sum_{h_k^m=0}^{\infty} g\,(h_k^m; p_k^m s_m) \ln g\,(h_k^m; p_k^m s_m) \quad \forall m \in M,\ k \in K_m^{UE} \tag{43}$$

where h_k^m is the path flow variable that follows a Poisson distribution with parameter $p_k^m s_m$. The base of the logarithm is not important as long as the same one is used consistently: change of base merely results in a rescaling of the entropy; here the natural logarithm is used. Given that the unconditional path flow follows a Poisson distribution independently of each other, the entropy corresponding to a strategic path choice p_k^m is

$$S\,(h_k^m) = p_k^m\, s_m\,[1 - \ln(p_k^m s_m)] + e^{-p_k^m s_m} \sum_{h_k^m=0}^{\infty} \frac{(p_k^m s_m)^{h_k^m} \ln h_k^m!}{h_k^m!} \forall m \in M,\ k \in K_m^{UE} \tag{44}$$

Note that the equilibrium path set for each O-D pair normally only contains a small number of candidate paths, leading to a relatively large $p_k^m s_m$. According to Evans et al. (1988), when $p_k^m s_m > 4$, the entropy function can already be well approximated by the following equation:

$$S\,(h_k^m) \approx \frac{1}{2} \ln(2\pi e p_k^m s_m) \quad \forall m \in M,\ k \in K_m^{UE} \tag{45}$$

Considering all the O-D pairs, the objective is to find a set of strategic path choice $[p_k^m]$ to maximize the entropy of the system

$$Max \rightarrow z_{entropy}\,([p_k^m]) = \sum_{m \in M} \sum_{k \in K_m^{UE}} \frac{1}{2} \ln(2\pi e p_k^m s_m) \tag{46}$$

subject to

$$\sum_{k \in K_m^{UE}} p_k^m = 1, \qquad \forall m \in M \tag{47}$$

$$p_k^m \geq 0, \qquad \forall k \in K_m^{UE},\ m \in M \tag{48}$$

$$\lambda_n = \sum_{m \in M} \sum_{k \in K_m^{UE}} p_k^m \delta_{n,k}^m\, s_m \qquad \forall n \in N \tag{49}$$

Proposition 3. *The strategic path choice* $[p_k^m]$ *is unique under the maximum entropy assumption.*

Equivalently, the objective function can be transformed into the following minimization problem with the same constraints:

$$min \rightarrow -z_{entropy}\,([p_k^m]) = -\sum_{m \in M} \sum_{k \in K_m^{UE}} \frac{1}{2} \ln(2\pi e p_k^m s_m) \tag{50}$$

The objective function is the sum of several convex functions, and all three constraints are also convex, therefore, a unique optimal solution exists. As the strategic link choice is the sum of several corresponding strategic path choices, it is also uniquely determined. The optimal solution can be obtained by various numerical methods such as the Newton's method and the gradient descent method.

2.5 Implementation algorithms of the model

Note that the identification of the equilibrium path set is required when solving the maximum entropy problem above. Although this equilibrium path set is unique, it is difficult to be obtained in practice due to computational precision limits. Hence, an approximation method similar to Larsson et al. (2001) is proposed here. In this method, all paths used in the all or nothing assignment procedure during the F-W algorithm are stored, until the strategic user equilibrium is reached. Then, the expected costs of all these paths are computed and those paths whose expected costs are within a tolerance threshold are saved as the approximated shortest paths, which will be used to solve the maximum entropy problem here. The tolerance threshold is defined as a proportion of the shortest path cost.

$$\frac{t\,(h_k^m) - t\,(h_k^{m*})}{t\,(h_k^{m*})} \le Tol \tag{51}$$

where $t(h_k^m)$ represents the expected cost of path k for O-D pair m, $t(h_k^{m*})$ represents the shortest path cost for O-D pair m. It must be mentioned that by doing so, only a subset of the equilibrium paths may be included, this error may be mitigated by setting a high relative gap (the difference between two consecutive iterations) for the F-W algorithm. The choice of tolerance threshold is also critical: low tolerance value may cause the optimization problem to be infeasible, high tolerance value may lead to inclusion of nonequilibrium paths. So the choice of the tolerance threshold should be carefully made.

The solution procedure of the model is demonstrated below:

Solution Algorithm:

Step 1: Initialization: Load free flow $[\lambda]_0$ to the network, and find the free flow travel cost on each link.

Step 2: All or Nothing Assignment: Based on the travel cost, find a set of expected link flows $[\lambda]_1 \rightarrow \min z([\lambda])$, subject to Equations (20) and (22), store the shortest path for each all or nothing assignment.

Step 3: Line search: Find the step size $\beta \rightarrow \min z(\beta[\lambda]_{n+1} + (1-\beta)[\lambda]_n)$, subject to $0 \le \beta \le 1$.

Step 4: Repeat Steps 2 and 3 until $\frac{[\lambda]_{n+1} - [\lambda]_n}{[\lambda]_{n+1}} \le \varepsilon$, where ε is the critical value which can be artificially set. In this step, the strategic user equilibrium is reached.

Step 5: Compute the expected cost of each stored path based on the equilibrium link cost.

Step 6: If $\frac{t(h_k^m) - t(h_k^{m*})}{t(h_k^{m*})} \le Tol$, save the path as a user equilibrium path.

Step 7: Solve the maximum entropy optimization program in Equations (46) to (49), a set of strategic path choice $[\ldots, p_k^m \ldots]$ is obtained. Various numerical methods, such as the Newton's method, may be applicable here.

Step 8: Calculate the strategic link choice from the strategic path choice.

Step 1 provides an initial feasible solution to start the algorithm, Step 2 builds the least cost paths set and loads corresponding traffic on these paths. In Step 3, a step size parameter β is sought to minimize the objective function, numerous methods are applicable here such as the Golden section method or Bisection method. Step 4 assesses the degree of convergence by computing the relative change in the expected link flow vector between iterations. Step 5 and Step 6 provide the equilibrium path set which will be used to determine their corresponding strategic path choice. Step 7 solves the optimization maximum entropy problem. Step 8 computes the strategic link choice from the strategic path choice. The F-W algorithm part can also be found in Cascetta (2009), the main difference is that in each all or nothing assignment iteration the shortest path will be stored in memory to determine the equilibrium paths.

3 NUMERICAL EXAMPLES

The link flow volumes (and travel times) may differ from the deterministic set of flows (and travel times) when demand variability is introduced. The strength of the proposed StrUE model is the incorporation of demand uncertainty into the user's decision-making process; therefore, the remainder of the analysis considers only stochastic demand conditions. Specifically, the demand follows a Poisson distribution with a prescribed parameter lambda. In the StrUE model, path proportions (and the corresponding link choice probabilities) are dependent on the demand distribution, and not any particular demand realization.

3.1 Monte Carlo simulation

To evaluate the StrUE model under demand variability, 100 randomly selected demand scenarios for a given demand distribution curve (with a prescribed Poisson parameter lambda) were generated through Monte Carlo sampling, then for each realized demand sample, 100 sets of multinomially distributed path flows are sampled based on the strategic path choices. As a result, we have 10,000 sets of sampled path flows and 10,000 sets of corresponding sampled link flows, which represents the unconditional distributions of link flow. These results derived from the Monte Carlo method will be indicated as simulated results, and those ones computed

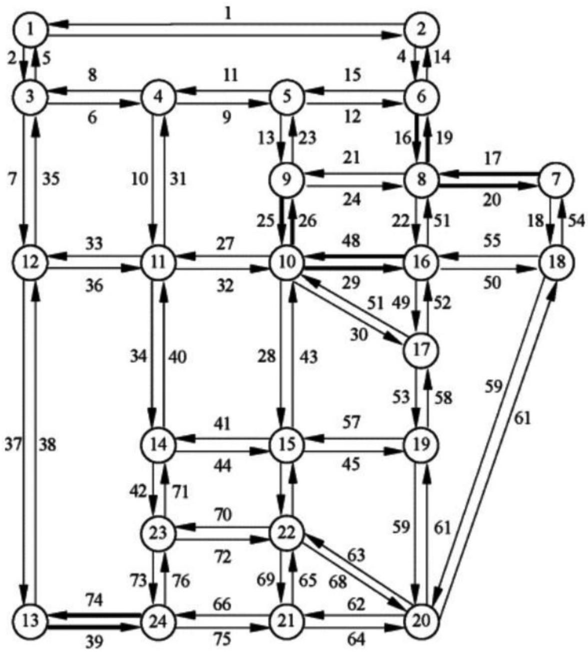

Fig. 1. The Sioux Falls network.

from the analytical equations will be indicated as estimated results. Based on the simulation the following performance measures are computed.

(1) Expected total system travel time (ETSTT) and standard deviation of total system travel time, (StdTSTT).
(2) Expected link travel times and standard deviation of link travel time.
(3) Expected link flow and standard deviation of link flow.

The mathematical expression of expected travel time on a link is given by the BPR function introduced in Equation (33). The F-W algorithm is implemented with the modified expected link travel time function to determine the strategic choice, and the corresponding link flow distributions, the relative gap for the F-W method is set to $1{*}e^{-5}$. The following results are based on the network depicted in Figure 1. The network has 24 nodes and 76 links. The capacity, free flow speed, length of each link, as well as other network attributes can be found in Bar-Gera (2012). The BPR parameters α and β are taken to be 0.15 and 4, respectively.

Figure 2 illustrates the probability distribution of flow on several randomly chosen links, the x-axis denotes the link flow and the y-axis represents the probability of observing the corresponding link flow. In total, there are 10,000 sets of sampled link flows which are generated from the 100 demand scenarios, these simulated link flows are plotted as a bar figure; the black curve represents the standard Poisson distribution with the parameter lambda derived from the strategic user equilibrium. It is demonstrated that the standard Poisson distribution curve is well approximated by the simulated results. This is further validated by a Chi-square test on these four links, the corresponding p-values are

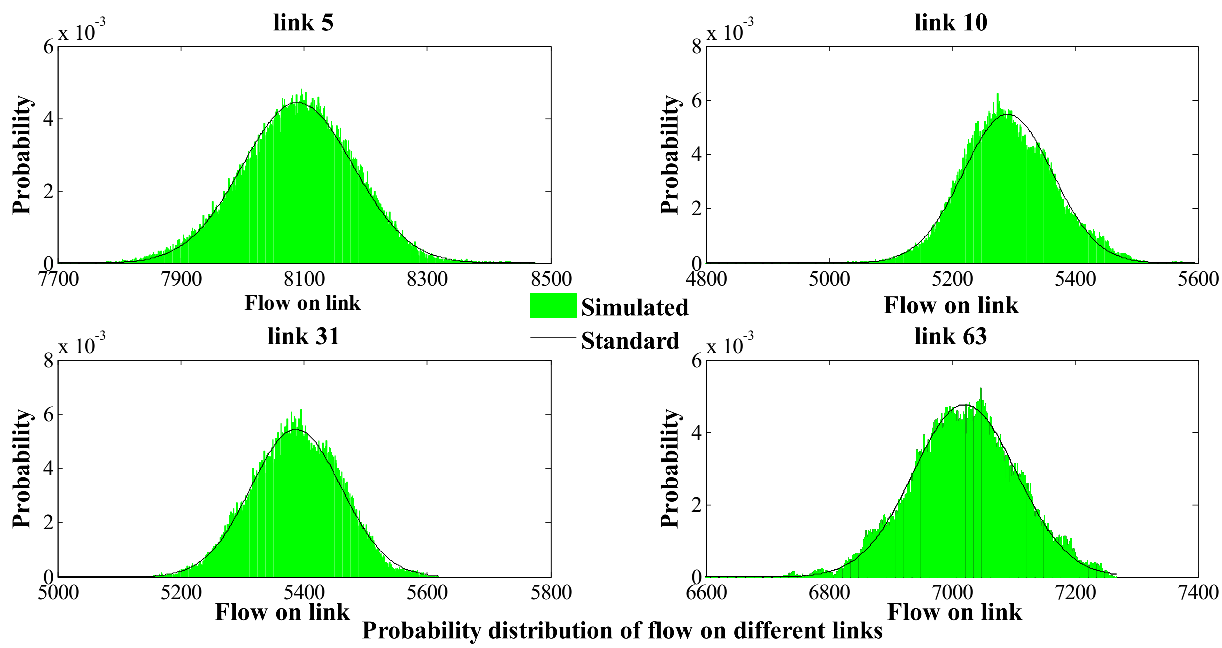

Fig. 2. The probability distribution of flow on several links.

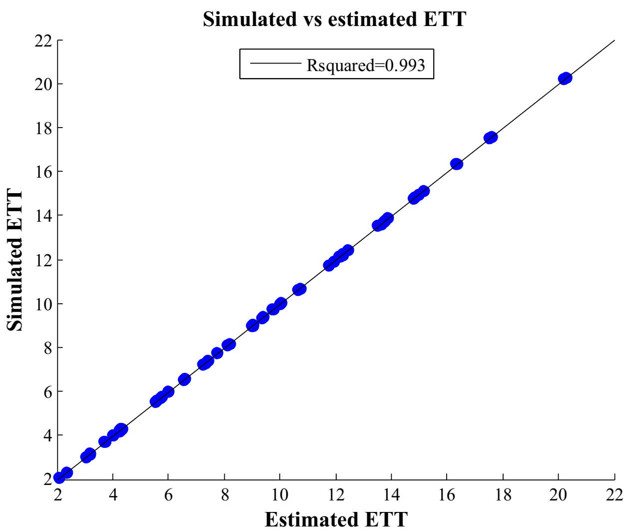

Fig. 3. Linear regression analysis of expected link travel time.

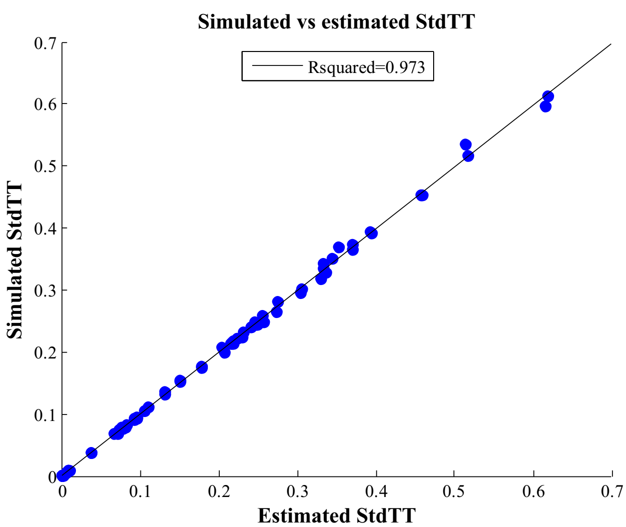

Fig. 4. Linear regression analysis of standard deviation of link travel time.

all equal to 1, which indicates that the simulated link flow is not significantly different from the corresponding standard Poisson distribution. Similar performance can be observed on other links too. Therefore, if each O-D demand follows a Poisson distribution independently, the corresponding unconditional link flow estimated by the model also follows a Poisson distribution, whose parameter is determined by users' strategic choice and demand distribution.

In Figures 3 and 4, the 10,000 sets of link flows are substituted in the BPR function to evaluate link travel times, which provide the expected link travel times (and respective standard deviation) for all the 76 links. In Figure 3, the x-axis represents the analytical travel time estimated from Equation (33) and the y-axis represents the expected link travel time computed from the simulated results. It is shown in Figure 3 that the analytically estimated expected link travel time closely approximates the simulated expected link travel times (as evident by the $R^2 = 0.99$).

One of the strengths of the StrUE model is its capability of estimating the link flow variation analytically. In Figure 4, the simulated standard deviations of link travel times are compared with the analytically derived values from Equation (34). Similar to the expected link travel time, a linear regression analysis is done on the estimated and simulated standard deviation of link travel time on all the 76 links. The R squared value is very close to 1 despite it is smaller than that of the expected link travel time. Note that the simulated standard deviation of link flow may deviate from the estimated ones when the expected travel time is large due to the sample size. Figure 4 demonstrates the model's ability to capture variability in travel time as a consequence of demand volatility.

Table 2
Network performance measures

Performance measures	*Estimated results*	*Simulated results*	*Relative error*
ETSTT	7,481,223	7,582,740	1.3%
StdTSTT	32,090.97	46,547.93	31.0%

Table 2 compares the system performance of ETSTT and StdTSTT of the network; the estimated results are computed from Equations (36) and (37). As demonstrated by the table, the simulated ETSTT is very close to the estimated ETSTT, as indicated by the relative error of just 1.3%. However, the StdTSTT is moderately different, due to the Monte Carlo sampling process: first 100 realized demands are sampled, then 100 multinomially distributed path flows are sampled based on each realized demand, so the path flows are actually correlated in each realized TSTT, however, the analytical expressions of StdTSTT are derived based on the unconditional path flows which are independent of each other. This inconsistency (correlation and independency) leads to the difference in estimated and simulated StdTSTT; therefore, the analytical expression of StdTSTT should be applied cautiously. Despite this, the ETSTT still presents a reliable approximation; in addition, the estimated expected link flows and the corresponding standard deviations satisfactorily match the simulated results, as shown in Figures 3 and 4.

Table 3
Path choices of MED and MEP for O-D pair (3-16)

Path index	*Path (represented by a sequence of nodes)*	*Path choice probability (MED)*	*Strategic path choice for (MEP)*
1	[3,4,5,6,8,16]	0.235	0.234
2	[3,4,5,9,8,7,18,16]	0.093	0.089
3	[3,1,2,6,8,16]	0.153	0.157
4	[3,4,5,9,10,16]	0.166	0.166
5	[3,4,5,6,8,7,18,16]	0.098	0.099
6	[3,4,5,9,8,16]	0.17	0.17
7	[3,1,2,6,8,7,18,16]	0.085	0.085

3.2 The most likely strategic link choice

Sometimes the aggregated link flows would not suffice when O-D-specific information are required, such as O-D matrix estimation, emission analysis and many other transportation applications. The maximum entropy provides a way to find the "most likely" strategic link choice. In the analysis, the relative gap (termination criterion) for the F-W method is set to $1{*}e^{-5}$, the tolerance threshold is set to 10%. In total, the F-W method provides 1,434 paths, among which 907 paths are identified as the equilibrium paths. To demonstrate the difference between the traditional maximum entropy method (the deterministic case, hereby referred to as MED) and the maximum entropy of probability distribution (where the entropy of path flow distribution is considered, hereby referred to as MEP), the results of these two methods are compared in Table 3. In MED, we first solve the Wardrop's user equilibrium, where all O-D demands are treated as deterministic variables. Then, the optimization program in Equations (39) to (42) is solved to provide the path choice probabilities. In MEP, we solve the optimization program in Equations (46) to (49) and the strategic path choice is obtained. In Table 3, O-D pair (3-16) is chosen to demonstrate the difference between MED and MEP, the demand for this O-D pair is 200. The differences in path choice can be seen in paths 1, 2, 3 and 5, and such difference may be more significant if only few paths are considered as user equilibrium paths. No ground truth is found yet to prove which method is better, but the results clearly indicate the necessity and importance of accounting for uncertainties in entropy.

Table 4 demonstrates the strategic link choice on link 24; the O-D pairs with no trips assigned to link 24 are not presented in the table. If the strategic link choice is one, it means all the users for that O-D pair will always choose this link. Note that the strategic link choice should not be greater than one. The importance of the strategic link choice lies in that it provides the link flow information disaggregated by O-D pairs. In real world, obtaining aggregated link flow is viable and efficient, and the strategic link choice can clearly present the "from-and-to" information on links.

Table 4
The unique strategic link choice for link 24

From origin	*To destination*	*Strategic link choice*
3	7	0.309
3	8	0.270
3	16	0.259
4	7	0.207
4	8	0.125
4	16	0.146
4	18	0.398
4	20	0.250
5	7	0.318
5	8	0.172
5	16	0.213
5	20	0.398
9	7	1.000
9	8	1.000
9	16	0.678
9	18	1.000
9	20	1.000
10	8	1.000
11	8	1.000
12	7	0.134
12	8	0.148
12	16	0.152
12	18	0.268
13	7	0.051
13	8	0.148

4 CONCLUSION

In this article, we proposed a model which extends the notion of the strategic user equilibrium originally proposed by Dixit et al. (2013) and Waller et al. (2013). The proposed model relaxed the assumption of proportional demand, hence improved the model fidelity. In this model, users equilibrate based on an expected condition as opposed to a deterministic cost. To capture this behavior, the model assumes that users rationally make their strategic link choice by considering all possible demand scenarios (all the O-D demands are independent of each other) in a known distribution. This strategic link choice is then followed regardless of the realized travel demand in any given scenario. Therefore, the state of equilibrium may not be observed on a given day. As such, the proposed model is illustrated to replicate the behavior of observed link travel time variability. In the proposed model, link flow distributions and users' strategic link choice are proved to be unique mathematically; network performance measures are given in analytical expression, which reduces the computation burden of network performance prediction. The efficiency and accuracy of these analytical expressions are demonstrated with a numerical example; the importance of accounting for probability distribution in entropy function is also presented. Therefore, this model accounts for the demand uncertainty and users' strategic choice while maintaining computation simplicity and tractability.

However, every model has its limitations. In this article, we assumed that O-D demand follows a Poisson distribution to ensure uniqueness, this forces the expected demand to be equal to the variance of demand, and this assumption may limit the applicability of the model.

Many possibilities are still to be explored under the model's framework. O-D demands may be assumed to follow some biparametric probability distributions if the uniqueness is guaranteed, which allows mean and variance to be different. In addition, we may account for capacity uncertainty by assuming the capacity also follows a certain kind of distribution. Finally, integrating this model into the O-D estimation problem would be a straight forward contribution.

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