A Maximum Likelihood Estimation of Trip Tables for the Strategic User Equilibrium Model

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ABSTRACT:

This paper proposes a novel framework to estimate trip tables for the strategic user equilibrium traffic assignment model. The proposed framework uses a bi-level estimation model, where the upper-level is a new maximum likelihood estimation method and the lower-level is the strategic user equilibrium assignment model which accounts for some aspects of day-to-day volatility in traffic flow. The maximum likelihood method proposed in this paper illustrates its ability to utilize information from day-to-day observed link flows in order to provide a unique estimation of the total trip demand distribution. This is accomplished by passing the total trip demand distribution to the strategic user equilibrium model to produce a set of link flow distributions which can then be compared to the link level observations. The mathematical proof demonstrates the convexity of the model. In addition, a numerical analysis is conducted on a test network to illustrate the efficiency of the proposed framework.

Keywords: strategic user equilibrium, O-D estimation, maximum likelihood

1. INTRODUCTION

Enhanced origin-destination (O-D) matrix estimation methodologies would be extremely useful for transportation planning. Traditionally, the O-D matrix is obtained from plate surveys, household surveys or roadside surveys. Such survey activities may be financially expensive for large size networks at frequent intervals, and usually suffer from limited response and sampling coverage. As an alternative, O-D matrix estimation provides a statistical approach for estimating or calibrating an O-D matrix from observed link flows and some prior knowledge of the O-D demand.

In this paper, we propose a framework that combines the maximum likelihood method for O-D matrix estimation and the strategic user equilibrium model (StrUE) for traffic assignment(*1*). This framework is hereby referred as MLStrUE. The StrUE model is defined such that "at strategic user equilibrium all used paths have equal and minimal expected cost". For each user present in a given demand scenario, their chosen route is followed regardless of the realized travel demand on a given day. Therefore the link flows will not result in an equilibrium state in any particular demand realization, but instead equilibrium exists stochastically over all demand realizations. The StrUE model was proposed to be able to capture the impact of day to day demand volatility on reliability, and eventually route choice. Therefore, apart from the O-D splits, a fundamental parameter that needs to be estimated is the variance in the total trip demand distribution.

An important aspect of the StrUE model is that the total trip demand is assumed to follow a certain statistical distribution; traditionally a lognormal distribution has been used (*2*, *3*). Under the assumption of a log-normally distributed demand, this paper focuses on estimating the demand distribution parameters. Note that other distributions can also be used if they do not change the convexity of the objective function. Only the distribution of the total trip demand needs to be estimated because for the StrUE model the O-D proportions are assumed to be fixed. Furthermore, for the StrUE model a log-normally distributed total trip demand allows for a closed form probability density function of the link flow, which can be shown to follow a lognormal distribution as well. The direct relationship between the link flow variables and the total demand in StrUE allows for the use of day-to-day observed link flows (which in turn provide actual link flow distributions) to calibrate the total demand distribution. In this study this calibration is accomplished by implementing the maximum likelihood estimation method, in which we maximize the joint probability of observing all the link flows within a time period.

A bi-level programming method is proposed to eliminate the impact of strongly biased initial estimates, where the upper level provides the total demand distribution to the StrUE model, and the StrUE model can provide link flow distributions to the upper level. A benefit of the proposed modelling framework includes the incorporation of actual day-to-day observed link flows and the corresponding distributions, instead of aggregated or averaged values. Additionally, the performance of the MLStrUE approach can be assessed based on the accuracy of its estimations for both expected link flows and link flow distributions, which are a direct output of the StrUE model.

The remainder of this paper includes a literature review of previous relevant research, presented in Section 2. Section 3 defines the mathematical model and includes a derivation for the analytical solution to the total demand estimation. Numerical analysis is demonstrated in Section 4; conclusions and future research are presented in Section 5.

2. LITERATURE REVIEW:

Although the traditional O-D matrix estimation mainly focused on statistical approaches based on loop counts, a wide range of methods have been explored in previous studies, including the generalized least square method(*4*, *5*), the maximum likelihood(*6*), bi-level programming approach(*7*) and maximum entropy(*8*). Generally the problem is to find an O-D matrix to optimize an objective function subject to a set of constraints. However, the problem is often challenging due to the number of observable links in a traffic network typically being much smaller than the number of O-D pair demands; this means that it may not be possible to obtain a unique solution from a single set of link counts alone. The problem was further extended to account for the stochastic nature of observed flows (*9, 10*). Recently, dynamic approaches were introduced to account for the time dependent characteristics in the network(*11, 12*). However, the application to large scale network and the computation complexity still remains a problem.

Among the research, relatively little attention has been paid to the higher order of the variables in a network, such as their variance and covariance that can potentially provide more constraints to the optimization problem. Cremer and Keller demonstrated that aggregating or averaging link count data collected over a sequence of time period may lose some important information.(*13*). Hazelton (*14*) proposed a weighted least squares method to account for the covariance of links, and assumed a parameter to explain the circumstances when the variance exceeds the mean if a Poisson distribution is used. Bell (*15*) proposed a maximum likelihood method and found the analytical solution to the covariance of O-D matrix by using a Taylor approximation. However, these research contributions still have some limitations in the assumptions. For example, the O-D demand was assumed to follow the Poisson or multinomial distribution, which stipulates certain relationships between the mean and variance of the O-D demand. In the MLStrUE, the O-D demand is assumed to follow a lognormal distribution, which allows the mean and variance of total demand to be independent of each other, and assures the non-negativity of the demand. In a well-constructed network, loop detectors can easily provide link counts on a day-to-day basis; therefore, it is important to consider the variation of link flows and the distribution of O-D total demand as effective information to calibrate the O-D trip matrix. The proposed MLStrUE framework estimates the distribution of the total O-D demand and thus significantly reduces computation complexity.

Estimation of the O-D trip matrix also requires a proper assignment model. When applying the assignment model to a large network, realism and computational complexity are both critical and must be equally considered to determine a model's practical applicability. Further, a major complication in transportation modelling is the ability to properly account for the inherent uncertainties regarding demand (*16, 17*) and capacity levels (*18, 19*). Additionally, as has been noted, uncertainty on these variables directly affects route choice behaviour (*20*). It is, therefore, necessary to incorporate these stochastic elements into models to ensure robust planning capabilities, but to do so in a manner that maintains computational tractability. The strategic user equilibrium (*1*) effectively accounts for the impact of demand uncertainty; the model relies on users minimizing their expected travel time based on the previous trip experiences in which they have gathered knowledge on demand (daily trips). The user's knowledge of each can be represented by a given distribution, with a known expected value and variance. Based on these known distributions, each user selects a travel route to minimize their expected travel time subject to Wardrop's UE conditions.

The contribution of this study can be highlighted as the following:

1) By assuming that the total demand follows a lognormal distribution, we exploit its positiveness. Additionally, unlike the Poisson distribution, it allows the variance and the mean of the total demand to be different.
2) We consider the day-to-day link flow variations and use a maximum likelihood method to relate this information to the total demand distribution.
3) We also apply the strategic user equilibrium model to account for the impact of variation in demand.

4) The StrUE model is used to provide link flows according to the estimated total demand distribution; multiple sets of link flows, thus the link flow distribution, is generated by sampling the total demand.

3. PROBLEM FORMULATION:

This section defines the mathematical concept of the MLStrUE framework. Table 1 explains the notations used in this paper.

TABLE 1 Summary of notations.

N	Link (index) set
K_{RS}	Path set
A	Node (index) set
p_n	Proportion of total demand on link *n*; $P = (p_1, \dots, p_n)$
t_n	Travel time on link *n*; **t** = $(\dots, t_n, \dots)$
t_{nf}	Free flow travel time on link *n*
c_k^{rs}	Travel time on path k connecting O-D pair r-s; $\mathbf{c}^{rs} = (\dots, c_k^{rs})$; $\mathbf{c} = (\dots, c^{rs}, \dots)$
q_{rs}	Fraction of total trips that are between O-D pair r-s; $\sum_{\forall rs} q_{rs} = 1$
T	Random variable for total trips with probability distribution $g(T)$
$g(T)$	Lognormal probability density function of the total demand
x_{ni}	Observed flow on link *n*, for day *i*.
l_n	Flow on link *n*.
v_k^{rs}	Proportion of flow on path k, connecting O-D pair r-s
C_n	The capacity on link *n*
$\delta_{n,k}^{rs}$	Indicator variable $\delta_{a,k}^{rs} = \begin{cases} 1\ if\ nis\ included\ in\ path\ k \\ 0\ otherwise \end{cases}$ $(\Delta^{rs})_{n,k} = \delta_{n,k}^{rs}$; $\Delta = (\dots, \Delta^{rs}, \dots)$
μ	Mean of the corresponding normal distribution, also called the location parameter for the lognormal distribution.
σ^2	Variance (= the second central moment) of the corresponding normal distribution, also called the scale parameter for the lognormal distribution.
σ	Standard deviation of the corresponding normal distribution
m_T	The expected total demand
v_T	Variance of the lognormal distribution
m_n	The expected link flow on link *n*.
v_n	The variance of link flow on link *n*.
s_T	The standard deviation of the total demand
s_n	The standard deviation of link flow on link *n*.
cov	Coefficient of variation, defined as the ratio of the standard deviation to the mean of a variable.

It is important to realize that μ and σ^2, which appear in the equations of the log-normal distribution, do not denote the mean and the variance of the log-normal distribution, but of the corresponding parameters of the normal distribution. The mean and the variance of the log-normal distribution are indicated in the following discussion by m and v.

The assignment map in the StrUE model is a vector noted as the link proportions, which is the proportion of the link flow to the total demand:

$$l_n = p_n T \qquad p_n \in P, n \in N \tag{1}$$

The link proportions are assumed fixed in the O-D matrix estimation problem; hence each link flow also follows a lognormal distribution if the total demand follows a lognormal distribution. It has been validated that numerical convolution of lognormal distributions is a distribution which follows the lognormal law with a fair approximation(*21*). The link flow distribution is related to the total demand distribution by:

$$m_n = p_n m_T \quad [2]$$
$$v_n = p_n^2 v_T \quad [3]$$

The parameters for the link flow distribution can be obtained by the definition:

$$\mu_n = ln\, m_n - \frac{1}{2} ln(1 + \frac{v_n}{m_n^2}) \quad [4]$$
$$\sigma_n^2 = ln(1 + \frac{v_n}{m_n^2}) \quad [5]$$

Substitute Eq.2 and Eq.3 into Eq.4 and Eq.5, we have the transformation of the total demand distribution to link flow distribution:

$$\sigma_n = \sigma_T \quad [6]$$
$$\mu_n = ln p_n + \mu_T \quad [7]$$

Since each link flow follows a lognormal distribution, the probability of observing x_n trips on link *n* is:

$$f(x_n) = \frac{1}{x_n \sigma_n \sqrt{2\pi}} e^{-\frac{(\ln x_n - \mu_n)^2}{2\sigma_n^2}} \quad n \in N \quad [8]$$

x_n-Observed flow on link *n*

The joint probability of observing a set of link flows can be obtained by the product of the probability density functions:

$$j(x_n) = \prod_1^n \frac{1}{x_n \sigma_n \sqrt{2\pi}} e^{-\frac{(\ln x_n - \mu_n)^2}{2\sigma_n^2}} \quad n \in N \quad [9]$$

Furthermore, we may collect more than one set of loop counts, namely the observed day-to-day link flows. It is therefore necessary to maximize the joint probability of observing all sets of link flows, in order to reduce the effect of noise and observation failure. Here the observed link flows are indicated by a *n*-by-*i* matrix, *n* is the number of links and *i* is the number of observations:

$$x_{ni} = \begin{bmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{ni} \end{bmatrix} \quad [10]$$

The maximum likelihood method here is to maximize the joint probability of observing all sets of link flows, which is given by the following equation:

$$j(x_n) = \prod_1^i \prod_1^n \frac{1}{x_{ni} \sigma_n \sqrt{2\pi}} e^{-\frac{(\ln x_{ni} - \mu_n)^2}{2\sigma_n^2}} \quad [11]$$

Conventionally, we maximize the logarithm of the joint probability, because taking log of the function won't change its convexity. By plugging in Eq.6 and Eq.7 into Eq.11 and changing the signs, the objective function becomes:

$$\min: J(t_j^*) = \sum_1^i \sum_1^n \ln(x_{ni}\sigma_T\sqrt{2\pi}) + \frac{(\ln\frac{x_{ni}}{p_n} - \mu_T)^2}{2{\sigma_T}^2} \quad [12]$$

Subject to: $\sigma_T > 0$

To prove the convexity of the objective function, we only need to show that for an arbitrary *x*, the function below is convex:

$$f(\mu_T, \sigma_T) = \sum_1^n \ln(x\sigma_T\sqrt{2\pi}) + \frac{(\ln x - \mu_T)^2}{2{\sigma_T}^2} \quad [13]$$

The Hessian matrix of $f(\mu_T, \sigma_T)$ can be found by taking second partial derivatives with respect to μ_j and σ_j:

$$H = \begin{bmatrix} {\sigma_T}^{-2} & 2\sigma_T^{-3}(lnx - \mu_T) \\ 2\sigma_T^{-3}(lnx - \mu_T) & \sigma_T^{-2} + 3\sigma_T^{-4}(lnx - \mu_T)^2 \end{bmatrix} > 0 \quad [14]$$

The Hessian is positive definite, hence the function is strictly convex. The sum of several convex functions is also a convex function, therefore we have proved that our objective function is strictly convex, the unique optimal solution is assured. The optimal solutions can be found by taking the first derivative with respect to mean and variance of total demand:

$$\mu_T = \frac{\sum_1^i \sum_1^n \ln\frac{x_{ni}}{p_n}}{ni} \quad [15]$$

$$\sigma_T^2 = \frac{\sum_1^i \sum_1^n (\ln\frac{x_{ni}}{p_n} - \mu_T)^2}{ni} \quad [16]$$

Assuming that the StrUE model represents the route choice behaviour, we can then formulate a bi-level programming problem, where the upper level is the maximum likelihood demand estimation problem; the lower level is the StrUE model, which has the objective function:

$$Minimize: z(f) = \sum_{n \epsilon N} \int_0^{f_n} \int_0^{\infty} \int_0^{\infty} t_n(p_n T)\, g(T) dT df \quad [17]$$

Subject to:

$$\sum_k v_k^{rs} = q_{rs} \qquad \forall k, r, s \quad [18]$$

$$v_k^{rs} \geq 0 \qquad \forall k, r, s \quad [19]$$

$$p_n = \sum_r \sum_s \sum_k v_k^{rs} \delta_{n,k}^{rs} \qquad \forall k, r, s \quad [20]$$

The fraction of the total demand between O-D pair *r*-*s*, namely q_{rs}, can be obtained from the prior estimates, i.e. household survey data, or field experiments. The link travel time function for the StrUE model is defined by the U.S. Bureau of Public Roads (*22*) cost function due to its widespread use in transport planning models:

$$t_n(flow) = t_{nf}\left[1 + \alpha t_{nf}\left(\frac{f_{nT}}{C_n}\right)^{\beta}\right] \quad [21]$$

where α and β are the parameters for the BPR function.

The objective functions of the upper and the lower levels are both strictly convex, therefore the model always has feasible solutions. A solution algorithm has been proposed to the bi-level programming:

Step 1: (Initialization) k=0. Start from the prior O-D matrix; obtain the fraction of total trips q_{rs} and initial values for the mean and the variance of the total demand. Produce a set of link proportions from the StrUE model. Note that q_{rs} will be kept invariant over the bi-level iterations while μ_T^k and σ_T^k will be calibrated.

Step 2: (Optimization) Substituting the link-flow proportion matrix P_k, solve the upper-level to obtain μ_T^k and σ_T^k of the total demand.

Step 3:(Simulation) Using μ_T^k and σ_T^k, apply the StrUE model to produce a new set of link flow proportions P_{k+1}.

Step 4: (Convergence test) Calculate the deviation between simulated and observed link flows, and the deviation between estimated and target O-D matrices. If stopping criterion is met, stop. After enough iteration, the results will always converge.

4. NUMERICAL RESULTS AND ANALYSIS:

The objective of the analysis is to test if the MLStrUE can effectively estimate the total demand distribution from day-to-day observed link flows. The estimated total demand distribution should closely approximate the actual total demand distribution; the link flow distribution produced by the StrUE model should also closely match the observed link flows. The main idea is to artificially determine the total demand distribution and generate random link flow samples accordingly. The MLStrUE will reproduce the desired total demand distribution from the random samples with perturbed prior estimates. The test should also reflect the scalability of the MLStrUE to networks of substantial complexity.

Numerical tests are conducted on the Sioux Falls network (24 nodes and 76 links). The network properties are pre-defined in (*23*) (see Fig.1). The notations used in this section are defined in Table 1. The O-D demand is specified as proportions of the total network demand, therefore the demand for a given O-D pair is the O-D proportion multiplied by the total demand. The BPR parameters α and β are equal to 0.15 and 4, respectively.

The observed link flows are generated by the following way:

Step 1: The true μ_T, σ_T are determined for the total demand.

Step 2: We implement the StrUE based on the total demand distribution and obtain a set of link proportions.

Step 3: We generate 100 samples of the total demand from the lognormal distribution using μ_T, σ_T as parameters and each sample total demand is assigned to the network using the pre-calculated link proportions.

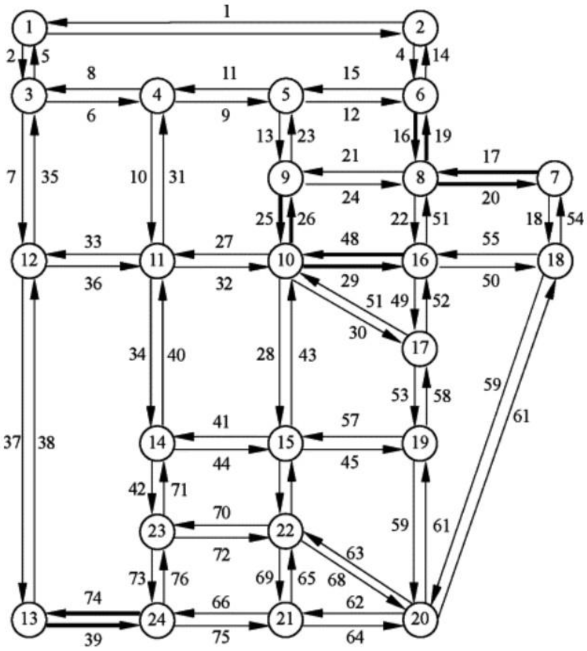

FIGURE 1 The Sioux Falls network.

The actual expected total demand of the Sioux Falls network is $m_A = 360600$, and the coefficient of variation cov is equal to 0.2, i.e. the standard deviation is 20% of the expected total demand. In Table 2, each scenario represents a different initial estimate of the total demand distribution.

TABLE 2 Different scenarios of initial estimation of O-D matrix

Scenario	Scenario description	m_T	s_T
1	$m_T = 0.8m_A$ and $cov = 0.1$	288480	28848
2	$m_T = 0.8m_A$ and $cov = 0.3$	288480	86544
3	$m_T = 1.2m_A$ and $cov = 0.1$	432720	43272
4	$m_T = 1.2m_A$ and $cov = 0.3$	432720	129816
5	$m_T = 1.5m_A$ and $cov = 0.1$	540900	54090
6	$m_T = 1.5m_A$ and $cov = 0.3$	540900	162270

In Fig.2 and Fig.3, the *x*-axis represents the number of iterations of the bi-level programming. In Fig.2, the *y*-axis represents the estimated expected total demand; In Fig.3, the *y*-axis represents the estimated standard deviation of the total demand. Both figures show that the estimated results converge to the actual ones in less than 3 iterations. This indicates that the MLStrUE's robust performance against biased initial estimates, and demonstrates the efficiency in arriving at convergence. In scenario 1 and 2, the estimated results of the first iteration in both figures are very different from the actual ones. This is mainly because the link proportions of the first iteration are obtained based on the initial estimates, and the initial estimates in scenarios 1 and 2 are very biased, and therefore the results of the first iteration will be inaccurate. If the initial estimates of total demand distribution are quite different from the actual distribution, the estimation of both expected total demand and standard deviation of total demand will be very inaccurate, therefore we have shown that applying the bi-level programming can reduce the impact of biased initial estimates.

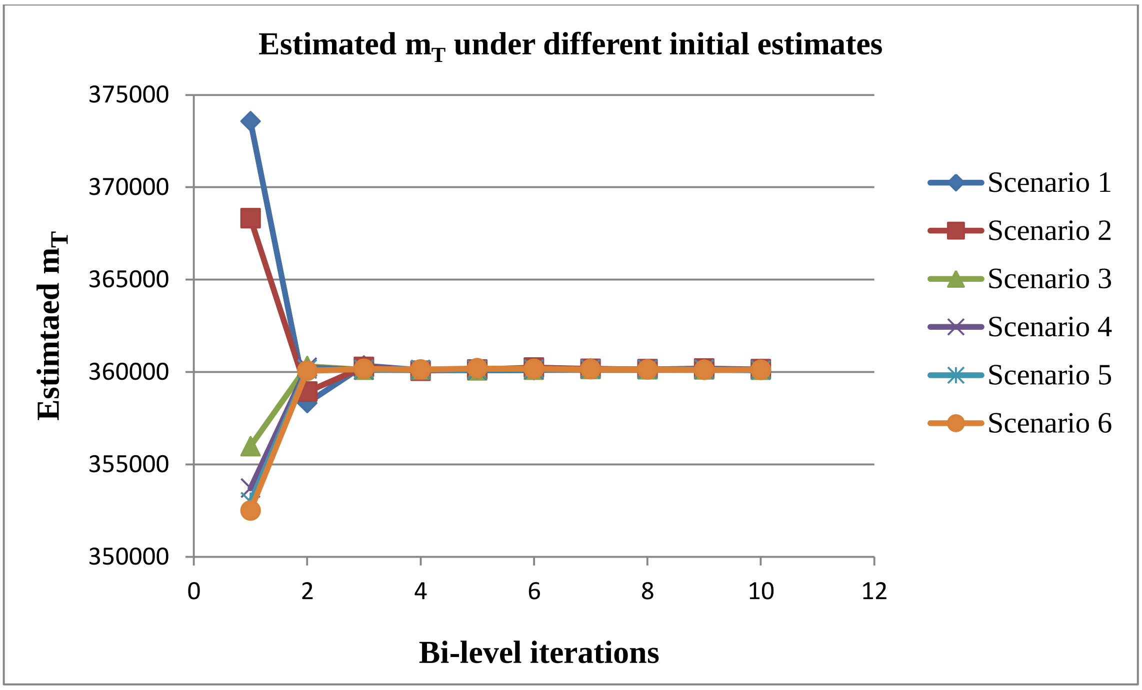

FIGURE 2 Estimated expected total demand under different scenarios of initial estimation; results of 10 bi-level iterations are presented.

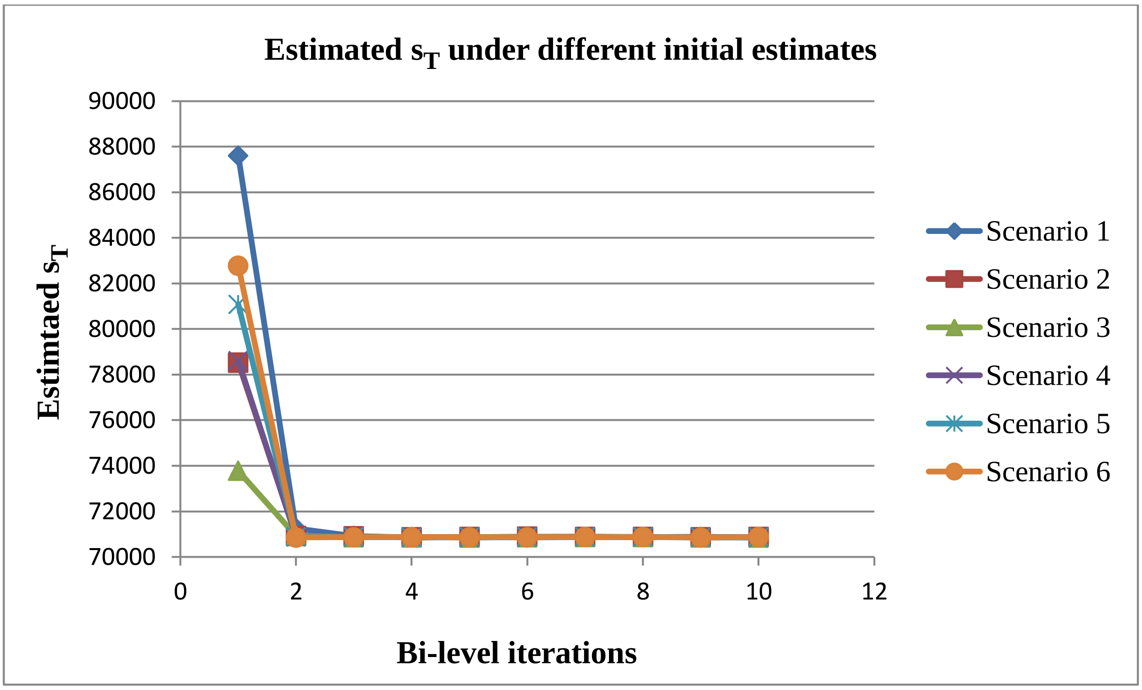

FIGURE 3 Estimated standard deviation of total demand under different scenarios of initial estimation; results of 10 bi-level iterations are presented.

Fig.4 demonstrates the performance of the MLStrUE at the link level; the *x*-axis indicates the actual expected link flow while the *y*-axis represents the estimated expected link flow. The estimated link flows are analytically produced by the StrUE model based on the total demand distribution after the convergence criterion has been met. The estimated expected link flows and the corresponding actual expected link flows are sorted from the smallest to the largest. The R squared value of the results is equal to 0.9837, which is very close to 1. This indicates that the estimated results closely approximate the actual expected link flows.

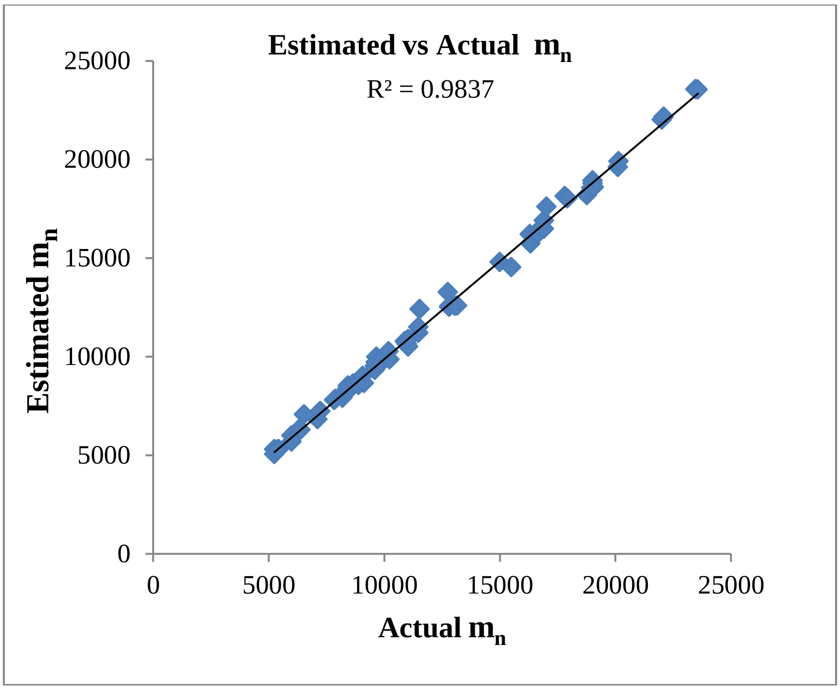

FIGURE 4 The estimated and actual expected link flow comparison, estimated results are produced by the StrUE model based on estimated demand distribution.

One of the strengths of the StrUE model is that it can produce the link flow variation. Since the total demand distribution is calibrated based on day-to-day observed link flows, it is therefore necessary to compare the estimated standard deviation of link flow to the actual one. In Fig.5, the estimated standard deviation of link flow is produced by the StrUE model based on the total demand distribution after the bi-level convergence criterion has been met. The *x*-axis denotes the actual standard deviation of link flow while the *y*-axis indicates the estimated one. It is illustrated in the figure that despite the fact that the R squared value is smaller than that of the expected link flow; the MLStrUE still provides relatively reliable estimation, however, if the standard deviation of link flow is very high, the estimated results may be more than 20% different from the actual ones.

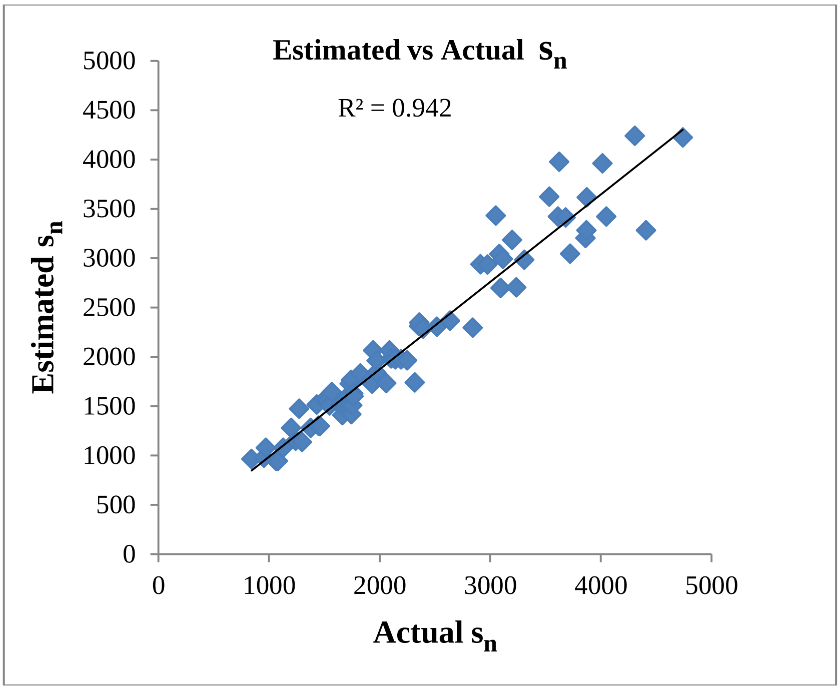

FIGURE 5 The estimated and actual standard deviation of link flow comparison, estimated results are produced by the StrUE model based on estimated demand distribution.

5. CONCLUSION:

This paper proposes a methodological framework (MLStrUE) to estimate the travel demand distribution (trip table) based on day-to-day observed link flows. The estimated total demand distribution maximizes the joint probability of observing all link flows. A bi-level programming method is also included to reduce the impact of biased initial estimates of the total demand distribution. A numerical analysis is conducted on a test network, and results for both the system level and the link level demonstrated robust performance of the MLStrUE framework. In the numerical experiment, the estimated mean and standard deviation of the total demand converged to the desired values regardless of the initial estimates after 2 or 3 iterations. Similarly, the link level analysis produced R squared values of 0.9837 and 0.942, for the expected value and standard deviation of link flows, respectively. Based on the results, the estimated link flow distribution closely approximates the actual link flow distribution, suggesting that the MLStrUE can calibrate the total demand effectively and efficiently.

One limitation of the MLStrUE is the assumption of perfect traffic loop count information. In this model, we generate loop counts by sampling from the results of the assignment model based on actual demand distribution, which may not reflect real world condition. In reality, the loop counts of some minor roads, or smaller regional roads might be missing in practice, and the failure of the loop detectors may also have impact on the results. The error may be reduced via statistical approaches such as outlier detection or noise analysis. Another limitation is that the prior estimates of demand proportions may influence the results. A real-world data set may be used in the future to validate the framework proposed here.

Future research will investigate the use of the covariance of loop counts, that is, if we have a large enough sample size, then the covariance matrix of link flows can be generated. This can potentially provide much more information than only mean and variance of link flows. Furthermore, the O-D demand may be assumed to follow a multivariate lognormal distribution, in this way the O-D demand is no longer aggregated as was the case with univariate lognormal distribution, possibly providing the covariance matrix of the link flows.

Since the OD estimation problem is a combination of a statistical optimization model and a traffic assignment model, an improvement in either process warrants further research.

6. ACKNOWLEDGEMENTS:

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7. REFERENCES:

(*1*) Dixit, V., L.M. Gardner, and S.T. Waller. *Strategic User Equilibrium Assignment Under Trip Variability*. in *Transportation Research Board 92nd Annual Meeting*. 2013.

(*2*) Wen, T., et al. *A Strategic User Equilibrium Model Incorporating Both Demand and Capacity 2 Uncertainty 3*. in *Transportation Research Board 93rd Annual Meeting*. 2014.

(*3*) Duell, M., et al. *Evaluation of a strategic road pricing scheme accounting for day-to-day and long term demand uncertainty*. in *Transportation Research Board 93rd Annual Meeting*. 2014.

(*4*) Cascetta, E., *Estimation of trip matrices from traffic counts and survey data: A generalized least squares estimator.* Transportation Research Part B: Methodological, 1984. **18**(4–5): p. 289-299.

(*5*) Bell, M.G.H., *The estimation of origin-destination matrices by constrained generalised least squares.* Transportation Research Part B: Methodological, 1991. **25**(1): p. 13-22.

(*6*) Spiess, H., *A maximum likelihood model for estimating origin-destination matrices.* Transportation Research Part B: Methodological, 1987. **21**(5): p. 395-412.

(*7*) Yang, H., et al., *Estimation of origin-destination matrices from link traffic counts on congested networks.* Transportation Research Part B: Methodological, 1992. **26**(6): p. 417-434.

(*8*) Fisk, C.S., *On combining maximum entropy trip matrix estimation with user optimal assignment.* Transportation Research Part B: Methodological, 1988. **22**(1): p. 69-73.

(*9*) Lo, H., N. Zhang, and W.H. Lam, *Estimation of an origin-destination matrix with random link choice proportions: a statistical approach.* Transportation Research Part B: Methodological, 1996. **30**(4): p. 309-324.

(*10*) Lo, H., N. Zhang, and W. Lam, *Decomposition algorithm for statistical estimation of OD matrix with random link choice proportions from traffic counts.* Transportation Research Part B: Methodological, 1999. **33**(5): p. 369-385.

(*11*) Frederix, R., F. Viti, and C.M.J. Tampère, *Dynamic origin–destination estimation in congested networks: theoretical findings and implications in practice.* Transportmetrica A: Transport Science, 2011. **9**(6): p. 494-513.

(*12*) Bierlaire, M. and F. Crittin, *An efficient algorithm for real-time estimation and prediction of dynamic OD tables.* Operations Research, 2004. **52**(1): p. 116-127.

(*13*) Cremer, M. and H. Keller, *A new class of dynamic methods for the identification of origin-destination flows.* Transportation Research Part B: Methodological, 1987. **21**(2): p. 117-132.

(*14*) Hazelton, M.L., *Some comments on origin–destination matrix estimation.* Transportation Research Part A: Policy and Practice, 2003. **37**(10): p. 811-822.

(*15*) Bell, M.G.H., *The Estimation of an Origin-Destination Matrix from Traffic Counts.* Transportation Science, 1983. **17**(2): p. 198-217.

(*16*) Kim, J., F. Kurauchi, and N. Uno, *Analysis of variation in demand and performance of urban expressways using dynamic path flow estimation.* Transportmetrica, 2009. **7**(1): p. 63-84.

(*17*) Bellei, G., et al., *A demand model with departure time choice for within-day dynamic traffic assignment.* European Journal of Operational Research, 2006. **175**(3): p. 1557-1576.

(*18*) Brilon, W., J. Geistefeldt, and M. Regler. *Reliability of freeway traffic flow: a stochastic concept of capacity*. in *Proceedings of the 16th International symposium on transportation and traffic theory*. 2005.

(*19*) Wu, X., P. Michalopoulos, and H.X. Liu, *Stochasticity of freeway operational capacity and chance-constrained ramp metering.* Transportation Research Part C: Emerging Technologies, 2010. **18**(5): p. 741-756.

(*20*) Uchida, T. and Y. Iida. *Risk assignment. A new traffic assignment model considering the risk of travel time variation*. in *Proceedings of the 12th International Symposium on the Theory of Traffic Flow and Transportation, Jul 21 - 23 1993*. 1993. Berkeley, CA, United states: Publ by Elsevier Science Publishers B.V.

(*21*) Fenton, L., *The sum of log-normal probability distributions in scatter transmission systems.* Communications Systems, IRE Transactions on, 1960. **8**(1): p. 57-67.

(*22*) U.S, *Department of Commerce. Bureau of Public Roads. Traffic Assignment Manual Urban Planning Division, US Department of Commerce, Washington, DC.* 1964.

(*23*) Bar-Gera, H., *Transportation test problems of Sioux falls network.* website http://www.bgu.ac.il/~bargera/tntp/, 2012.