Journal of the Operational Research Society (2015) 66, 664–673

www.palgrave-journals.com/jors/

Solving mixed-integer robust optimization problems with interval uncertainty using Benders decomposition

Sauleh Siddiqui[1*], Steven A Gabriel[2] and Shapour Azarm[3]

[1]Johns Hopkins University, Baltimore, MD, USA; [2]Department of Civil Engineering, Applied Mathematics & Statistics, and Scientific Computation Program, University of Maryland, College Park, MD, USA; and [3]Department of Mechanical Engineering, Applied Mathematics & Statistics, and Scientific Computation Program, University of Maryland, College Park, MD, USA

Uncertainty and integer variables often exist together in economics and engineering design problems. The goal of robust optimization problems is to find an optimal solution that has acceptable sensitivity with respect to uncertain factors. Including integer variables with or without uncertainty can lead to formulations that are computationally expensive to solve. Previous approaches for robust optimization problems under interval uncertainty involve nested optimization or are not applicable to mixed-integer problems where the objective or constraint functions are neither quadratic, nor linear. The overall objective in this paper is to present an efficient robust optimization method that does not contain nested optimization and is applicable to mixed-integer problems with quasiconvex constraints ($\leqslant$ type) and convex objective funtion. The proposed method is applied to a variety of numerical examples to test its applicability and numerical evidence is provided for convergence in general as well as some theoretical results for problems with linear constraints.

Journal of the Operational Research Society (2015) **66**(4), 664–673. doi:10.1057/jors.2014.41

Published online 23 April 2014

Keywords: robust optimization; mixed-integer; Benders decomposition; interval uncertainty; quasiconvex function

1. Introduction

Optimization problems in economics and engineering often involve mixed-integer decision variables and uncontrollable variations or uncertainties. Optimal, feasible solutions might end up being infeasible for particular realizations of uncertainty factors. Manufacturing errors, measurement problems, and uncertainty in inputs are examples of sources for these variations. Combined with having integer decision variables, finding global solutions to such problems can be computationally expensive.

In this paper, an approach for robust optimization for linear, quadratic, convex, and quasiconvex mixed-integer programs is developed by applying a worst-case analysis using the Benders decomposition method. Whereas in the standard Benders decomposition method (Benders, 1962) there is only one set of complicating variables solved for in the master problem, mixed-integer robust optimization problems have two sets of complicating variables: the integer variables and the uncertainty variables. Hence, two distinct master problems with accompanying optimal value functions α_y and α_u, respectively will be introduced in this paper. Intervals with a nominal point (user- or problem-defined) are used to represent uncertainty and no probability distribution is presumed. A real-world design situation is reflected in this paper, for example, when information about uncertain factors during the early stages of a design process is often limited. This paper is an extension of Siddiqui *et al* (2011) that provided numerically verifiable solutions to continuous robust optimization problems. This current paper provides theoretical foundations to the related paper and extends it to mixed-integer robust optimization problems.

This paper's approach (hereafter referred to as the *Robust Benders method*) has been tested and verified with several optimization problem examples, 13 of which are presented in this paper. A comprehensive review of the literature was conducted and to the best of our knowledge, tractable solutions to these problems, examples of mixed-integer robust optimization problems with convex objective functions and quasiconvex constraints, have not been reported.

In addition to considering quasiconvex constraints, the robust optimization problems solved by the proposed Robust Benders method also require a separable (the function can be written as the sum of a function involving only the integer variables and another involving only the continuous variables) objective

**Correspondence: Sauleh Siddiqui, Department of Civil Engineering, Department of Applied Mathematics & Statistics, Johns Hopkins University, 3400 N Charles Street, Baltimore, MD 21218, USA.*

function. This paper presents theoretical evidence that the Robust Benders method works for separable convex objective functions and numerical evidence that it works for separable nonlinear objective functions, both with quasiconvex constraints. This is more general than only considering a linear objective function in the problem as reported in the literature (eg, Soyster, 1973; Balling *et al*, 1986; Ben-Tal and Nemirovski, 2002) or quadratic (eg, Li *et al*, 2011) as well as other versions involving convex programs (eg, Ganzerli and Pantelides, 1999) or linearization to solve the problem (eg, Balling *et al*, 1986). Note that many of the aforementioned studies cannot be readily extended to handle mixed-integer variables. Exceptions include Li *et al* (2011) that gives exact solutions to convex quadratic programs and only approximate solutions to non-convex quadratic programs. Bertsimas and Sim (2004a, b) provide solutions to mixed-integer linear programming (MILP) robust optimization problems, as well as a linear approximation scheme for other programs. Solving robust binary linear programs has also been discussed in Atamturk (2006) and Wu (2011). Other integer solution techniques with uncertainty (Zhu and Sherali, 2009; Zeghal *et al*, 2011) are also present in the literature.

There has been an abundance of literature modifying Benders decomposition (Benders, 1962) to solve various types of optimization problems, including integer programs (Beale, 1965). A method to obtain robust solutions for a supply chain planning problem using Benders decomposition was provided by Poojari *et al* (2008). Saito and Murota (2007) described a method to apply Benders decomposition to solve linear, mixed-integer, robust optimization problems with ellipsoidal uncertainty. However, this approach only works for linear problems, while the Robust Benders method of this paper is applicable for the class of problems solvable by the original Benders decomposition method. It is important to note that the method of this paper is not an extension to the method provided in Saito and Murota (2007), but a novel application of Benders decomposition not present in the literature. Finally, Montemanni (2006) applied a Benders algorithm to a specific robust spanning tree problem, while Ng *et al* (2010) applied it to a specific semiconductor allocation problem that had uncertainty.

In the next section, the background terminology and problem definition are described along with supporting theory. Then, in Section 3, a detailed formulation of the proposed Robust Benders method is presented. Section 4 provides numerical and engineering examples from the literature that highlight the differentiating characteristics of this method. Some concluding remarks will be provided in Section 5.

2. Background terminology and problem definition

Table 1 describes the terminology used in this paper.

The goal in robust optimization is to optimize the objective function with respect to continuous decision variables x and discrete decision variables y, satisfying all constraints and ensuring the objective variation is kept within an acceptable range Δf_0, while accounting for uncertainty. Specifically, this paper considers robust optimization problems of the form:

Table 1 Definition of terms

Symbol	*Interpretation*
x	Vector of continuous decision variables
y	Vector of integer decision variables
f	Scalar Objective function to be minimized
$g_j(x, y, \hat{x}, \hat{y})$	Scalar Constraint functions of the form '$\leqslant 0$'
$\Delta x, \Delta y$	Maximum deviations of uncertainty from nominal values
$\hat{x}, \hat{y}$	Uncertainty variables: Deviations from nominal values of uncertain variables: $\hat{x} \in [-\Delta x, \Delta x], \hat{y} \in [-\Delta y, \Delta y]$
Δf_0	User specified tolerance for acceptable variation in objective function under uncertainty

$$
\begin{aligned}
&\min_{x,y} f(x, y, 0, 0) \\
\text{s.t.} \quad & \frac{f(x, y, \hat{x}, \hat{y}) - f(x, y, 0, 0)}{\Delta f_0} \leqslant 1, \\
& \quad \forall \hat{x} \in ([-\Delta x, \Delta x] \cap R^{n_u}), \quad \forall \hat{y} \in ([-\Delta y, \Delta y] \cap Z^{m_u}) \\
& g_j(x, y, \hat{x}, \hat{y}) \leqslant 0 \quad j = 1, \ldots, J, \\
& \quad \forall \hat{x} \in ([-\Delta x, \Delta x] \cap R^{n_u}), \quad \forall \hat{y} \in ([-\Delta y, \Delta y] \cap Z^{m_u}) \\
& x \in R^n, y \in Z^m
\end{aligned} \tag{1}
$$

For ease of notation, it will not be explicitly mentioned unless necessary that y and $\hat{y}$ are integers. However, since worst-case uncertainty is being considered, a continuous range for $\hat{y} \in [-\Delta y, \Delta y]$ will be used since it is more conservative than $\hat{y} \in [-\Delta y, \Delta y] \cap Z^{m_u}$, hence, consistent with the robust optimization worst-case uncertainty framework. In the next few paragraphs, we define terms used in the paper.

2.1. Objective robustness

For a candidate point (x^c, y^c), *objective robustness holds if the inequality*

$$\frac{f(x^c, y^c, \hat{x}, \hat{y}) - f(x^c, y^c, 0, 0)}{\Delta f_0} \leqslant 1 \tag{2}$$

is satisfied for all $\hat{x} \in [-\Delta x, \Delta x]$ *and for all* $\hat{y} \in [-\Delta y, \Delta y]$.

Thus, this inequality ensures that the maximum positive objective function variation stays below a certain predetermined maximal amount Δf_0 when presented with deviations in uncertain variables. Note that there is no absolute value around the left-hand side of the inequality in (2) because a lower objective function value for a particular realization of the uncertainty is actually a favourable outcome, and as such does not need to be constrained.

2.2. Feasibility robustness

For a candidate solution (x^c, y^c) *if*

$$g_j(x^c, y^c, \hat{x}, \hat{y}) \leqslant 0 \qquad \forall j = 1, \ldots, J \tag{3}$$

is satisfied for all $\hat{x} \in [-\Delta x, \Delta x]$ *and for all* $\hat{y} \in [-\Delta y, \Delta y]$, *then feasibility robustness holds.*

Note that inequality (2) is just another constraint, so it can be easily incorporated into inequality (3) when stating formulation (1). From this point on, inequality (2) will not be stated separately in any formulation but will be assumed to be incorporated into inequality (3). Hence, whenever a property will be attributed or required of a constraint function $g_j(x^c, y^c, \hat{x}, \hat{y})$, we assume that it applies to $(f(x, y, \hat{x}, \hat{y}) - f(x, y, 0, 0))/\Delta f_0$ as well. For example, whenever we require $g_j(x^c, y^c, \hat{x}, \hat{y})$ to be convex, it will be assumed that $(f(x, y, \hat{x}, \hat{y}) - f(x, y, 0, 0))/\Delta f_0$ is required to be convex as well. A more detailed description on objective robustness can be found in Li *et al* (2006). Since (2) has now been included in (3), the following formulation will be followed in the paper. Note that the uncertainty variables have also been removed from the objective function, as they are extraneous when (2) has been incorporated into (3) (recall that we do not minimize over uncertainty variables, only decision variables). For this paper, we assume that the constraints and objective function are such that a solution exists without uncertainty, that is, a solution exists when at nominal: $\hat{x} = \hat{y} = 0$, so for ease of notation we have removed them explicitly fixed to 0 from the objective function.

$$\begin{aligned} &\min_{x,y} f(x, y) \\ \text{s.t.}\quad & g_j(x, y, \hat{x}, \hat{y}) \leqslant 0 \qquad j = 1, \ldots, J \\ & x \in R^n, y \in Z^m, \hat{x} \in R^{n_u}, \hat{y} \in Z^{m_u} \\ & \forall \hat{x} \in [-\Delta x, \Delta x], \forall \hat{y} \in [-\Delta y, \Delta y] \end{aligned} \tag{4}$$

2.3. Robust point

A point that satisfies feasibility robustness for (4) *is a robust point. The set of robust points for* (4) *is denoted* S_R.

2.4. Globally optimal robust

For (4), *a globally optimal robust solution* (x^*, y^*) *is a robust point that is optimal* $(f(x^*, y^*) \leqslant f(x, y), \forall (x, y) \in S_R)$

The goal is to choose values of x and y such that the formulation (4) gives an optimal solution regardless of the values of $\hat{x}$ and $\hat{y}$. Since this paper only considers the worst-case analysis, the method aims to get the 'worst' values of $\hat{x}$ and $\hat{y}$ for (4) These are called 'globally interval-optimal' values, as defined next.

2.5. Globally interval-optimal

For a particular candidate solution (x^c, y^c) *and set of constraint functions* g_j; $j = 1, \ldots, J$, *a globally interval-optimal point is defined as a point* $(\hat{x}^c, \hat{y}^c) \in ([-\Delta x, \Delta x], [-\Delta y, \Delta y])$ *such that* $g_j(x^c, y^c, \hat{x}, \hat{y}) \leqslant \max_j g_j(x^c, y^c, \hat{x}^c, \hat{y}^c)$ *for all realizations of* $\hat{x} \in [-\Delta x, \Delta x]$ *and for all* $\hat{y} \in [-\Delta y, \Delta y]$.

The point $(\hat{x}^c, \hat{y}^c) \in ([-\Delta x, \Delta x], [-\Delta y, \Delta y])$ in the definition above is a particular value of the $(\hat{x}, \hat{y})$ such that the constraints in (4) attain their global maximum value for that particular candidate value of uncertainty. For continuous g, a globally interval-optimal point exists for each candidate. Certain assumptions need to be made before proceeding, which are stated below.

Assumption 1 *The constraints* g_j *in* (4), *for any fixed value of uncertainty* $\hat{x}, \hat{y}$, *form a convex, compact, nonempty feasible region over* x.

Assumption 2 *A globally optimal robust solution to* (4) *exists.*

The above assumptions deal with the existence of a robust solution. Since robust optimization problems often occur in engineering design and economics applications, the problem structure dictates that all variables and uncertainties can be bounded by either physical quantities and/or economic limitations. The convexity and compactness of the feasible region is required for the theoretical foundations, and can be relaxed for specific applications if the aforementioned properties are satisfied locally. The following Lemma 1 relates the definitions of a globally interval-optimal point and robust point. The theoretical foundation will be based on interval-optimality, and Lemma 1 helps it proceed from robust optimization to finding interval-optimal points.

Lemma 1 *A candidate solution* (x^c, y^c) *for problem* (4) *is a robust point if and only if its globally interval-optimal point* $(\hat{x}^c, \hat{y}^c) \in ([-\Delta x, \Delta x], [-\Delta y, \Delta y])$ *is such that* $\max_j g_j(x^c, y^c, \hat{x}^c, \hat{y}^c) \leqslant 0$.

Proof If (x^c, y^c) is a robust point, then it must be true that $\max_j g_j(x^c, y^c, \hat{x}, \hat{y}) \leqslant 0$ for all realizations of $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$. Hence, this implies that for the associated globally interval-optimal point$(\hat{x}^c, \hat{y}^c)$, $\max_j g_j(x^c, y^c, \hat{x}^c, \hat{y}^c) \leqslant 0$ as $(\hat{x}^c, \hat{y}^c) \in ([-\Delta x, \Delta x], [-\Delta y, \Delta y])$. For the other side of the if and only if argument, suppose the associated globally interval-optimal point has $\max_j g_j(x^c, y^c, \hat{x}^c, \hat{y}^c) \leqslant 0$. Then by the definition of globally interval-optimal, $\max_j g_j(x^c, y^c, \hat{x}, \hat{y}) \leqslant 0$ for all realizations of $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$ which implies that (x^c, y^c) is a robust point. □

Benders decomposition involves defining an auxiliary function (of only the complicating variables) and then approximating that function using Benders cuts. There are two sets of complicating variables in this paper: the uncertainty variables and the integer variables. Along with minimizing with respect to the uncomplicating variables (x), the objective is to minimize with respect to the integer complicating variables (y) while making sure that the uncertainty variables $(\hat{x}, \hat{y})$ take on interval-optimal values. From Lemma 1, we can see then that the optimization problem needs to be maximized (verified by Lemma 2 later) with respect to the uncertainty variables. Hence, two auxiliary functions (resulting in two master problems for Benders decomposition) will be defined as below:

$$\begin{aligned} &\alpha_u(\hat{x}, \hat{y}) = \min_{x,y} f(x, y) \\ \text{s.t.}\quad & g_j(x, y, \hat{x}, \hat{y}) \leqslant 0 \qquad j = 1, \ldots, J \\ & x \in R^n, y \in Z^m \end{aligned} \tag{5}$$

$$\begin{aligned} &\alpha_y(y) = \min_{x} f(x, y) \\ \text{s.t.}\quad & g_j(x, y, \hat{x}, \hat{y}) \leqslant 0, \qquad j = 1, \ldots, J \\ & \hat{x} = \hat{x}_{\text{fixed}}, \hat{y} = \hat{y}_{\text{fixed}} \\ & x \in R^n \end{aligned} \tag{6}$$

For further results, another definition and assumption are required as stated below.

2.6. Worst-case uncertainty

A worst-case uncertainty value $(\hat{x}^{\text{wc}}, \hat{y}^{\text{wc}}) \in ([-\Delta x, +\Delta x], [-\Delta y, +\Delta y])$ *for optimization problem* (4) *is such that when* $\hat{x} = \hat{x}^{\text{wc}}, \hat{y} = \hat{y}^{\text{wc}}$ *is fixed in* (4), *any solution of* (7) *below yields a globally optimal-robust solution.*

$$\begin{aligned} &\min_{x,y} f(x, y) \\ \text{s.t.}\quad & g_j(x, y, \hat{x}, \hat{y}) \leqslant 0, \qquad j = 1, \ldots, J \\ & g_j(x, y, 0, 0) \leqslant 0, \qquad j = 1, \ldots, J \\ & \hat{x} = \hat{x}^{\text{wc}}, \hat{y} = \hat{y}^{\text{wc}} \\ & x \in R^n, y \in Z^m \end{aligned} \tag{7}$$

Note that a worst-case uncertainty value differs from a globally interval-optimal value of uncertainty in that a worst-case uncertainty value does not have an associated predetermined variable x but it is associated with a globally optimal robust solution *after* solving (7). But it is trivial to note that a worst-case uncertainty value of $\hat{x}, \hat{y}$ is a globally interval-optimal point of uncertainty for some globally optimal robust solution. For the examples in this paper, globally interval-optimal values and worst-case uncertainty values coincide, which may not always be the case. Consider the following simple example of a robust optimization problem (without integer variables for simplicity):

$$\begin{aligned} &\min f(x) = -x_1 - 2x_2 \\ \text{s.t.}\quad & g_1 \equiv (1+\hat{x}_1)x_1 + (1+\hat{x}_2)x_2 \leqslant 8 \\ & g_2 \equiv (-2+\hat{x}_3)x_1 + (1+\hat{x}_4)x_2 \leqslant 5 \\ & g_3 \equiv (-1+\hat{x}_5)x_1 + (-3+\hat{x}_6)x_2 \leqslant -10 \\ & \forall \hat{x}_i \in [-0.1, 0.1], \qquad i = 1, \ldots, 6 \end{aligned}$$

Note that parameter uncertainty has been introduced in the constraints of the problem. Realize also that, for example, if $\hat{x}_1 = \hat{x}_2 = 0.1$ in the first constraint, then if x_1 and x_2 satisfy the following inequality

$$(1+0.1)x_1 + (1+0.1)x_2 \leqslant 8$$

then x_1 and x_2 also satisfy

$$\begin{aligned} &(1+\hat{x}_1)x_1 + (1+\hat{x}_2)x_2 \leqslant 8 \\ &\forall \hat{x}_i \in [-0.1, 0.1], \qquad i = 1, 2 \end{aligned}$$

Hence, this 'trick' can be applied to all parameters and we can get an optimization problem which will give us a robust solution. In this case, $\hat{x}_i = 0.1, \quad i = 1, ..., 6$ is a globally interval-optimal value for any candidate solution x. It is also the worst-case uncertainty value for this optimization problem, as when fixed in the following formulation, it defines the set of robust points S_R (feasible region of the following optimization problem) and helps obtain the globally optimal robust solution (the solution to the following optimization problem).

$$\begin{aligned} &\min f(x) = -x_1 - 2x_2 \\ \text{s.t.}\quad & g_1 \equiv (1+0.1)x_1 + (1+0.1)x_2 \leqslant 8 \\ & g_2 \equiv (-2+0.1)x_1 + (1+0.1)x_2 \leqslant 5 \\ & g_3 \equiv (-1+0.1)x_1 + (-3+0.1)x_2 \leqslant -10 \end{aligned}$$

The solution to this robust optimization problem can be found by looking at the corner points of the robust feasible region which gives $x_1=1$, $x_2=69/11$ (approximately 6.27), $f(x)=-149/11$ (approximately -13.54), with associated worst-case uncertainty value $\hat{x}_i = 0.1, \quad i = 1, ..., 6$.

The next assumption[1] is required for the theoretical background of the Robust Benders method presented in this paper. Again, given that engineering design and economics applications are often bounded with non-empty feasible regions, this assumption becomes reasonable to apply to any practical application.

Assumption 3 *A worst-case uncertainty value exists for robust optimization problem* (4) *and is a globally interval-optimal point for a globally optimal robust solution* (x*, y*).

[1]Note that Assumption 3 ensures that finding worst-case uncertainty values enables us to find a globally optimal robust solution.

From this point on, Assumptions 1–3 are enforced throughout the paper. Note that Assumption 3 is valid when the objective function is strictly convex and the constraint functions are strictly quasiconvex (proof in Lemma 3 later), or when constraint functions are monotone (trivial). If we have a unique globally interval-optimal point, we automatically get the existence of a worst-case uncertainty value (hence the requirement of strict convexity and quasiconvexity). The following lemma shows a property of the new auxiliary function associated with uncertainty variables (α_u), which connects a globally optimal robust point to its globally interval-optimal point. This will later be used in modifying Benders decomposition to obtain solutions to robust optimization problems.

Lemma 2 *Under Assumptions 1–3, let* (x^*, y^*) *be a globally optimal robust solution and* $(\hat{x}^*, \hat{y}^*)$ *an associated globally interval-optimal point for problem* (4)*. If (i)* $\alpha_u(\hat{x}^*, \hat{y}^*) \geqslant \alpha_u(\hat{x}, \hat{y})$ *for all realizations of* $\hat{x} \in [-\Delta x, \Delta x]$ *and for all* $\hat{y} \in [-\Delta y, \Delta y]$*, then (ii)* $\alpha_u(\hat{x}^*, \hat{y}^*) = f(x^*, y^*)$.

Proof Since (x^*, y^*) is a globally optimal robust point, it is automatically a robust point so by Lemma 1, $\max_j g_j(x^*, y^*, \hat{x}, \hat{y}) \leqslant 0$ for all realizations of $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$. Note that the value $\alpha_u(\hat{x}^*, \hat{y}^*)$ was calculated by minimizing $f(x, y)$ while fixing $\hat{x} = \hat{x}^*, \hat{y} = \hat{y}^*$ in (5). Since (x^*, y^*) is in the feasible region for (5) and by Assumption 2, a solution always exists to (5), $\alpha_u(\hat{x}^*, \hat{y}^*) \leqslant f(x^*, y^*)$. The next step will show with the help of a contradiction argument that $\alpha_u(\hat{x}^*, \hat{y}^*) \geqslant f(x^*, y^*)$. Suppose that $\alpha_u(\hat{x}^*, \hat{y}^*) < f(x^*, y^*)$. By the statement of this lemma, $\alpha_u(\hat{x}^*, \hat{y}^*) \geqslant \alpha_u(\hat{x}, \hat{y})$ for all $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$. Using (5) by fixing $\hat{x} = \hat{x}^*, \hat{y} = \hat{y}^*$, let x', y' (dependent on $\hat{x}^*, \hat{y}^*$) be a solution to the minimization problem in (5) such that $\alpha_u(\hat{x}^*, \hat{y}^*) = f(x', y')$. Then (i) implies $f(x', y') \geqslant \alpha_u(\hat{x}, \hat{y})$ for all $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$. By our contradictory assumption, this also implies $f(x^*, y^*) > \alpha_u(\hat{x}^*, \hat{y}^*) = f(x', y') \geqslant \alpha_u(\hat{x}, \hat{y})$ which simplifies to $f(x^*, y^*) > \alpha_u(\hat{x}, \hat{y})$ for all $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$. Note that the condition $f(x^*, y^*) > \alpha_u(\hat{x}, \hat{y})$ for all $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$ violates Assumption 3. By Assumption 3, there exists a worst-case uncertainty value, that is, there exists a $(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) \in ([-\Delta x, \Delta x], [-\Delta y, \Delta y])$ such that $f(x^*, y^*) = \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$. But this would imply $\alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) > \alpha_u(\hat{x}, \hat{y})$ for all $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$ which is a contradiction. Hence, this contradiction shows that $\alpha_u(\hat{x}^*, \hat{y}^*) \geqslant f(x^*, y^*)$. Combining the two inequalities $\alpha_u(\hat{x}^*, \hat{y}^*) \geqslant f(x^*, y^*)$ and $\alpha_u(\hat{x}^*, \hat{y}^*) \leqslant f(x^*, y^*)$ gives $\alpha_u(\hat{x}^*, \hat{y}^*) = f(x^*, y^*)$. □

Note that $\alpha_u(\hat{x}, \hat{y}) = \min_y \alpha_y(y) = \min_{x,y} f(x, y)$. The next two theorems form the basis of the Robust Benders method. The first shows a particular characteristic of a worst-case value of uncertainty. The second shows that a particular characteristic of an uncertainty variable value can be used to find a globally optimal robust solution.

Theorem 1 *Let a worst-case value of uncertainty for* (4) *be* $(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$*. Then,* $\alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) \geqslant \alpha_u(\hat{x}, \hat{y})$ *for all realizations of* $\hat{x} \in [-\Delta x, \Delta x]$ *and for all* $\hat{y} \in [-\Delta y, \Delta y]$.

Proof Let (x^*, y^*) be a globally optimal robust solution to (4). Then, by definition, $\alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) = f(x^*, y^*)$. Problem (4) has the same objective function as (5) but the feasible region of (4) is a subset of the feasible region of (5). Therefore, for any fixed $\hat{x}, \hat{y}\, f(x^*, y^*) \geqslant \alpha_u(\hat{x}, \hat{y})$. Therefore, $\alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) \geqslant \alpha_u(\hat{x}, \hat{y})$ for all realizations of $\hat{x} \in [-\Delta x, \Delta x]$ and for all $\hat{y} \in [-\Delta y, \Delta y]$. □

Theorem 2 *Suppose there exists a unique uncertainty value vector* $(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}})$ *for which* $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \geqslant \alpha_u(\hat{x}, \hat{y})$ *for all realizations of* $\hat{x} \in [-\Delta x, \Delta x]$ *and* $\hat{y} \in [-\Delta y, \Delta y]$*. Then, a solution to optimization problem* (8) *will be a globally optimal robust solution to problem* (4).

$$
\begin{aligned}
&\min_{x,y} f(x, y) \\
\text{s.t.}\quad & g_j(x, y, \hat{x}, \hat{y}) \leqslant 0, j = 1, \dots, J \\
& \hat{x} = \hat{x}^{\mathrm{c}}, \hat{y} = \hat{y}^{\mathrm{c}} \\
& x \in R^n, y \in Z^m
\end{aligned} \tag{8}
$$

Proof Let $(x^{\mathrm{c}}, y^{\mathrm{c}})$ be a solution to (8), which implies that $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) = f(x^{\mathrm{c}}, y^{\mathrm{c}})$ by (5). By Assumption 3, there exists a worst-case uncertainty value $(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) \in ([-\Delta x, \Delta x], [-\Delta y, \Delta y])$ such that $f(x^*, y^*) = \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$, where (x^*, y^*) is a globally optimal robust solution to (4). Since (x^*, y^*) is a solution to (4), it is also feasible to (8) as the feasible region for (4) is a subset of the feasible region for (8). Hence, $f(x^{\mathrm{c}}, y^{\mathrm{c}}) \leqslant f(x^*, y^*)$, which implies $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \leqslant \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$. We now claim that $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \leqslant \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}}) \Rightarrow (\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) = (\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$. We will prove this claim by contradiction. Suppose the above claim is not true, that is, $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \leqslant \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$ but $(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \neq (\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$. That would imply, by the premise in the theorem statement that if $(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \neq (\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$ then $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) > \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$, which contradicts $\alpha_u(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \leqslant \alpha_u(\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$. Hence, the premise $(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) \neq (\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$ is incorrect and we conclude $(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}}) = (\hat{x}^{\mathrm{wc}}, \hat{y}^{\mathrm{wc}})$, which implies $(\hat{x}^{\mathrm{c}}, \hat{y}^{\mathrm{c}})$ is a worst-case uncertainty value. By the definition of a worst-case uncertainty value, (8) gives a globally optimal robust solution. □

The purpose of Theorem 2 is that if the following optimization problem[2] (9) has a unique solution, that solution can be used to find a solution to (4).

$$
\begin{aligned}
&\max_{\hat{x},\hat{y}} \alpha_u(\hat{x}, \hat{y}) \\
\text{s.t.}\quad & \hat{x} \in [-\Delta x, \Delta x], \hat{y} \in [-\Delta y, \Delta y], \hat{x} \in R^{n_u}, \hat{y} \in Z^{m_u}
\end{aligned} \tag{9}
$$

[2]Note that the function $\alpha_u(\hat{x}.\hat{y})$ is not known in closed form but will be later shown to be approximated using a variation of Benders cuts.

3. Formulation of approach: solving mixed-integer robust optimization problems with convex objective and quasiconvex constraints

Theorem 2 shows that finding globally interval-optimal points can help us obtain an optimal robust solution. For quasiconvex functions, we know that the interval-optimal points will lie on one of the endpoints (and so will the globally interval-optimal points), which follows directly from the definition below.

3.1. Quasiconvex function

A function q: $\Omega \to R$ *defined on a convex subset* Ω *is said to be quasiconvex if for all* $x, y \in \Omega$ *and* $\lambda \in [0, 1]$ $q(\lambda x + (1-\lambda)y) \leqslant \max\{q(x), q(y)\}$. *In particular, if a function* $q(z)$:$[-\Delta z, \Delta z] \to R$ *is quasiconvex over a vector interval* $[-\Delta z, \Delta z] \in R^M$, *then for all* $z \in [-\Delta z, \Delta z]$, $q(z) \leqslant \max\{q(\Delta z_m)\}$ *where* $\Delta z_m \in (\{\Delta z_i\}_{i=1}^{M} \cup \{-\Delta z_i\}_{i=1}^{M})$.

Realize that all linear functions as well as convex functions are quasiconvex. Considering the constraint functions of (4) to be quasiconvex greatly simplifies the problem, and allows an easier use of Benders decomposition. Problem (4) is now rewritten as problem (10) and problem (9) is rewritten as problem (11). For purposes of notation, let the endpoints of the vector interval be denoted by $(V^x)_1$, $(V^x)_2$, $(V^x)_3, \ldots, (V^x)_{2^{n_u}}$. Each endpoint vector $(V^x)_k$, $k = 1, \ldots, 2^{n_u}$ is defined such that each of its elements $(V_i^x)_k$ is either Δx_i or $-\Delta x_i$, that is, $(V_i^x)_k \in \{-\Delta x_i, \Delta x_i\}$ for $i = 1, \ldots, n_u$. The notation for $(V_i^y)_k$ is similar.[3]

$$
\begin{aligned}
&\min_{x,y} f(x,y) \\
\text{s.t.}\quad & g_j(x,y,\hat{x},\hat{y}) \leqslant 0, \qquad j = 1,\ldots,J \\
& g_j(x,y,0,0) \leqslant 0, \qquad j = 1,\ldots,J \\
& x \in R^n, y \in Z^m, \hat{x} \in R^{n_u}, \hat{y} \in Z^{m_u}, \\
& \forall \hat{x} \in \{(V^x)\}_{k=1}^{2^{n_u}} \forall \hat{y} \in \{(V^y)_k\}_{k=1}^{2^{m_u}}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\max_{\hat{x},\hat{y}} \alpha_u(\hat{x},\hat{y}) \\
\text{s.t.}\quad & \hat{x} \in \{(V^x)_k\}_{k=1}^{2^{n_u}}, \hat{y} \in \{(V^y)_k\}_{k=1}^{2^{m_u}}
\end{aligned} \tag{11}
$$

Lemma 3 *Under Assumptions* 1–2, *Assumption* 3 *holds for problem* (10) *when f is strictly convex and g are strictly quasiconvex.*

Proof A worst-case uncertainty value is a globally interval-optimal point by definition. By Assumption 2, let (x^*, y^*) be a globally optimal robust solution to (10) and $(\hat{x}^*, \hat{y}^*)$ its associated interval-optimal point. We will prove the existence of a worst-case uncertainty value via contradiction. Assume $(\hat{x}^*, \hat{y}^*)$ is not a worst-case uncertainty value for (10), that is, solving (10) with $(\hat{x}^*, \hat{y}^*)$ fixed as in (7) yields a non-robust solution (x^c, y^c) for which $f(x^c, y^c) \leqslant f(x^*, y^*)$. We can write $g_j(x,y,\hat{x},\hat{y}) = g_j(x+\hat{x}, y+\hat{y})$ and find $(\hat{x}^p, \hat{y}^p)$ such that $(x^* + \hat{x}^p, y^* + \hat{y}^p)$ is a convex combination of (x^*, y^*) and (x^c, y^c) and thus $g(x^* + \hat{x}^p, y^* + \hat{y}^p, \hat{x}^*, \hat{y}^*) < \max(g(x^*, y^*, \hat{x}^*, \hat{y}^*), g(x^c, y^c, \hat{x}^*, \hat{y}^*)) \leqslant 0$ implying $(x^* + \hat{x}^p, y^* + \hat{y}^p)$ is a feasible solution to (10). Owing to strict convexity $f(x^* + \hat{x}^p, y^* + \hat{y}^p) < \max(f(x^*), f(x^c))$. But this violates optimality of x^* and thus our assumption that there can be a non-robust solution is violated. Therefore, Assumption 3 holds. □

Numerical evidence indicates that α_y convex in y and α_u concave in $\hat{x}, \hat{y}$ results in the Robust Benders decomposition method converging to a solution. There is a larger class of functions than all linear programs for which these conditions are valid. In particular, this class (for which α_y convex in y and α_u concave in $\hat{x}, \hat{y}$) encompasses the class of all optimization problems for which Benders decomposition converges. Indeed, for many engineering applications as well as numerical examples, local convexity of α_y and local concavity of α_u can be sufficient (Conejo *et al*, 2006). The next two theorems provide a more general setting for α_y to be convex in y and $\alpha_u$is convex. The algorithm provided works when $\alpha_u$is quasiconvex.

Theorem 3 *The function* α_y *defined in* (6) *is convex when the objective function of* (4) *is convex and separable into x and y functions and the constraint functions are quasiconvex such that the feasible region of* (4) *is convex.*

Proof Note that f is separable into the two variables x and y, that is, $f(x, y) = f_x(x) + f_y(y)$. The subscript with the objective and constraint functions denotes this separability. The feasible region of (5), for any fixed values of $\hat{x}$ and $\hat{y}$, is convex based on Assumption 1. Fix $\hat{x}$ and $\hat{y}$ to any specific values $\hat{x}_{\text{fixed}}$ and $\hat{y}_{\text{fixed}}$, respectively. Hence, the feasible region of (6) is a subset of the feasible region of (5), restricted to fixed values of y. If there are no feasible solutions or one feasible solution then the proof is vacuously true. Consider two feasible solutions of problem (6), $s^1 = (x^1, y^1)$ and $s^2 = (x^2, y^2)$ in such a way that for the variables y^1 and y^2, the associated solutions from (6) are x^1 and x^2, respectively, with associated objective function values $\alpha_y(y^1) = f_x(x^1)$ and $\alpha_y(y^2) = f_x(x^2)$, respectively. Let $s^3 = (x^3, y^3)$ be the convex combination of s^1 and s^2, that is, for $\lambda \in [0, 1]$, $s^3 = \lambda s^1 + (1-\lambda)s^2$. Now consider the value of the objective function at s^3. We have $f_x(x^3) = f_x(\lambda x^1 + (1-\lambda)x^2) \leqslant \lambda f_x(x^1) + (1-\lambda)f_x(x^2)$ because f_x is convex. This implies $f_x(x^3) \leqslant \lambda\alpha_y(y^1) + (1-\lambda)\alpha_y(y^2)$. But using y^3 as a fixed value in (6), the optimization problem (6) can be solved to obtain $\alpha_y(y^3)$. Let this solution to (6) be signified by x^* and hence $\alpha_y(y^3) = f_x(x^*)$. Because of an optimality argument, $f_x(x^*) \leqslant f_x(x^3)$. Hence, $\alpha_y(y^3) \leqslant \lambda\alpha_y(y^1) + (1-\lambda)\alpha_y(y^2)$ and this shows that $\alpha_y$is convex. □

[3] Since only endpoints are considered,$\hat{y}$ does not need to be distinguished from $\hat{x}$ and the same procedures can be applied. However, if Δy is not integer, then taking the first integer value less than Δy will ensure an optimal robust solution.

Theorem 4 *The function α_u defined in* (5) *(with y relaxed to be real) is convex when the objective function of* (4) *is convex and the constraint functions are quasiconvex such that the feasible region of* (4) *is convex.*

Proof Consider two points $s^1 = (\hat{x}^1, \hat{y}^1)$, $s^2 = (\hat{x}^2, \hat{y}^2)$ and let $s^3 = (\hat{x}^3, \hat{y}^3)$ be the convex combination of s^1 and s^2, that is, for $\lambda \in [0, 1]$, $s^3 = \lambda s^1 + (1-\lambda)s^2$. Let $\alpha(\hat{x}^1, \hat{y}^1) = f(x^1, y^1)$, $\alpha(\hat{x}^2, \hat{y}^2) = f(x^2, y^2)$, and $(x^3, y^3) = \lambda(x^1, y^1) + (1-\lambda)(x^2, y^2)$ for $\lambda \in [0, 1]$ which implies $f(x^3, y^3) \leqslant \lambda f(x^1, y^1) + (1-\lambda)f(x^2, y^2)$ by the convexity of f. Then, since g is quasiconvex, we have $g(x^3, y^3, \hat{x}^3, \hat{y}^3) \leqslant \max(g(x^1, y^1, \hat{x}^1, \hat{y}^1), g(x^2, y^2, \hat{x}^2, \hat{y}^2)) \leqslant 0$. Hence, $(x^3, y^3)$is in the feasible region of $\alpha(\hat{x}^3, \hat{y}^3)$ which implies α_u is convex by an optimality argument $\alpha(\hat{x}^3, \hat{y}^3) \leqslant f(x^3, y^3) \leqslant \lambda f(x^1, y^1) + (1-\lambda)f(x^2, y^2) \leqslant \lambda\alpha(\hat{x}^1, \hat{y}^1) + (1-\lambda)\alpha(\hat{x}^2, \hat{y}^2)$. □

Unfortunately standard Benders cuts cannot be used to approximate quasiconvex functions that are not convex. Our advantage in a robust optimization setting with quasiconvex constraints as in problem (10) is that we only need good approximations to the functions at the endpoints. Approximations of the function α_u are not needed in between the endpoints, as the function attains its maximum at the endpoints. Figure 1 shows the idea behind these new cuts. The very top horizontal cut (labelled as 'Cut 0') is an upper bound set for α_u as would normally occur in Benders decomposition. The numbers next to the cuts show the order of the cuts made in the iterative process. At iteration *it* a new Benders cut added to the master problem looks like the following:

$$\alpha_u \leqslant f\left(x_{it}^{sol}, y_{it}^{sol}, \hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right) + \frac{f\left(x_{it}^{sol}, y_{it}^{sol}, \hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right) - f\left(x_{it-1}^{sol}, y_{it-1}^{sol}, \hat{x}_{it-1}^{sol}, \hat{y}_{it-1}^{sol}\right)}{\left(\hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right)^T - \left(\hat{x}_{it-1}^{sol}, \hat{y}_{it-1}^{sol}\right)^T} \times \left((\hat{x}, \hat{y})^T - \left(\hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right)^T\right) \tag{12}$$

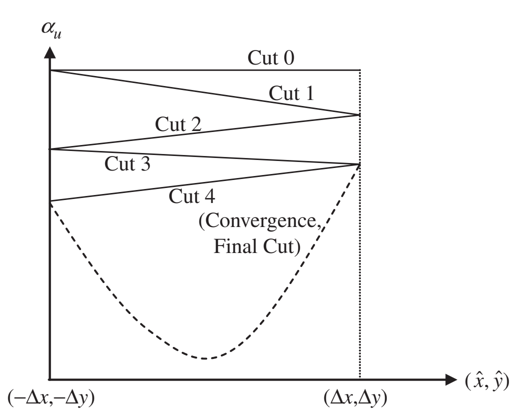

Figure 1 The Robust Benders cuts to estimate the maximum endpoint of α_u.

This cut is one way to approximate the function around the endpoints and see which one has a larger value of α_u as shown in Figure 1.

The following describes the Robust Benders decomposition method for a separable convex objective function and quasiconvex constraints forming a convex feasible set.

Set iteration counter (*it*) to 0. Pick a small positive constant for tolerance (*tol*).

Step 1: Set iteration counter (*it*) to $it = it + 1$. The variables $(\hat{x}, \hat{y})$ are *complicating* variables since we fix them in the subproblems. Hence, the original master problems will be[4]:

$$\begin{aligned} &\min_{\alpha_y, y} \alpha_y + f_y(y) \\ \text{s.t.}\quad & y_{\text{low}} \leqslant y \leqslant y_{\text{high}} \\ & \alpha_y \geqslant \alpha_y^{\min} \end{aligned} \tag{13}$$

$$\begin{aligned} &\max_{\alpha_u, \hat{x}, \hat{y}} \alpha_u \\ \text{s.t.}\quad & -\Delta x \leqslant \hat{x} \leqslant \Delta x \\ & -\Delta y \leqslant \hat{y} \leqslant \Delta y \\ & \alpha_u \leqslant \alpha_u^{\max} \end{aligned} \tag{14}$$

The bounds on α_y,α_u are user-defined depending on the problem. Solving the above problem gives $\alpha_y = \alpha_y^{it}$, $\alpha_u = \alpha_u^{it}$ and $y = y_{\text{fixed}}^{it}, \hat{x} = \hat{x}_{\text{fixed}}^{it}, \hat{y} = \hat{y}_{\text{fixed}}^{it}$.

Step 2: Fix the values of the complicating variables, and then solve the following subproblem as in the standard Benders decomposition method.

$$\begin{aligned} & w = \min_x f_x(x) \\ \text{s.t.}\quad & g_j(x, y, \hat{x}, \hat{y}) \leqslant 0, \qquad j = 1, \dots, J \\ & y = y_{\text{fixed}}^{it} \quad (Dual : \lambda^{it}) \\ & \hat{x} = \hat{x}_{\text{fixed}}^{it} \\ & \hat{y} = \hat{y}_{\text{fixed}}^{it} \end{aligned} \tag{15}$$

Step 3: Check for convergence. Set $z_{sub} = w$ and $z_{mas1} = \alpha_y^{it}$, $z_{mas2} = \alpha_u^{it}$ If the difference $(|z_{sub} - z_{mas1}|)/z_{sub} \leqslant tol$ and $(|z_{sub} - z_{mas2}|)/z_{sub} \leqslant tol$ then stop.

Step 4: Add Benders cuts to the respective master problems. To problem (13), add a standard Benders cut

$$\alpha_y \geqslant f\left(x_{it}^{sol}, y_{it}^{sol}, \hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right) + \lambda^{it}\left(y - y_{it}^{sol}\right) \tag{16}$$

While for the master problem (14) add the Robust Benders cut (12).

[4]Note that the complicating variables do not appear in the objective function for the master problem (14).

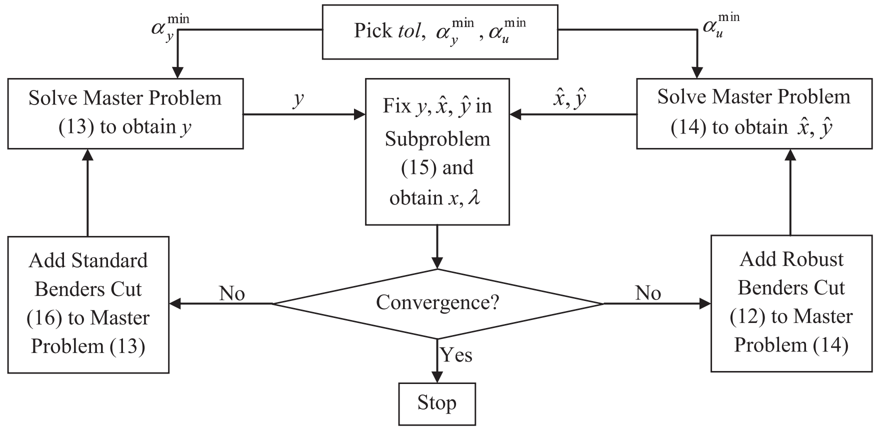

Figure 2 The Robust Benders cuts decomposition method.

Step 1 *(returned)*: Let *IT* be the number of iterations completed. Solve the following master problems after adding the Benders cuts

$$\min_{\alpha_y, y} \alpha_y + f_y(y)$$

$$\text{s.t.} \quad y_{\text{low}} \leqslant y \leqslant y_{\text{high}}$$

$$\alpha_u \geqslant f\left(x_{it}^{sol}, y_{it}^{sol}, \hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right) + \lambda^{it}\left(y - y_{it}^{sol}\right)$$

$$\text{for } it = 1, \ldots, IT$$

$$\alpha_y \geqslant \alpha_y^{\min} \tag{17}$$

$$\max_{\alpha_u, \hat{x}, \hat{y}} \alpha_u$$

$$\text{s.t.} \quad -\Delta x \leqslant \hat{x} \leqslant \Delta x$$

$$-\Delta y \leqslant \hat{y} \leqslant \Delta y$$

$$\alpha_u \leqslant f\left(x_{it}^{sol}, y_{it}^{sol}, \hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right)$$

$$+ \frac{f\left(x_{it}^{sol}, y_{it}^{sol}, \hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right) - f\left(x_{it-1}^{sol}, y_{it-1}^{sol}, \hat{x}_{it-1}^{sol}, \hat{y}_{it-1}^{sol}\right)}{\left(\hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right)^T - \left(\hat{x}_{it-1}^{sol}, \hat{y}_{it-1}^{sol}\right)^T}$$

$$\times \left((\hat{x}, \hat{y})^T - \left(\hat{x}_{it}^{sol}, \hat{y}_{it}^{sol}\right)^T\right)$$

$$\alpha_u \leqslant \alpha_u^{\max}$$

$$\text{For} \quad it = 1, \ldots, IT \tag{18}$$

Return to Step 2 and proceed in this manner until convergence is met.

The flowchart in Figure 2 gives an outline of the algorithm. One of the important factors of this algorithm is that there are two requirements for convergence, both of which must be met to have an optimal robust solution. Nonconvergence of the α_y master problem implies that the solution is not integer-optimal for y and nonconvergence of the α_u master problem implies that the solution might not be robust. Note that Gabriel *et al* (2009) provide a heuristic to test when the α functions are convex as well as a workaround for when they are not. A general convergence proof for this algorithm is part of ongoing research and we believe this section has provided a good basis for this theory. The numerical results in Section 4 provide further support to this approach.

4. Numerical results

The examples in Table 2 show the applicability of the algorithm to various types of robust optimization problems. The first six numerical problems consist of two robust mixed-integer linear programs, two robust mixed-integer quadratic programs, and two robust programs with quasiconvex constraints. Examples 1 and 2 have been taken from Li *et al* (2011), examples 3 and 4 from Siddiqui *et al* (2011) and examples 5 and 6 from Hock and Schittkowski (1980). An engineering example of the design of a heat exchanger is also presented and forms the seventh test problem. Detailed formulations as well as further characteristics of the solution are in Siddiqui *et al* (2011). Note that while the theoretical foundation required there be a unique worst-case uncertainty value, the last three examples did not have a unique worst-case uncertainty value but were still able to provide locally optimal robust solutions similar to Siddiqui *et al* (2011). The next six text problems are from Floudas *et al* (1999). Solutions were checked by a simple uniform discretization of the uncertainty range (each point separated by 0.01). Tolerance (*tol*) was set to 0.00001 for all examples. Table 2 describes the results obtained from the numerical test problems. All solutions were shown to be optimal robust, and the function calls and solution time were also at a reasonable level. An important point to note is that the solution time and function calls increase at not more than an order of magnitude when going from linear and quadratic problems to problems with quasiconvex constraints. Note that the Heat Exchanger example is solved using an additional heuristic as described in Siddiqui *et al* (2011) to ensure the constraints are quasiconvex before applying the algorithm in this paper.

Table 2 Description of test problems

Source	*# of variables (Cont., Integer)**	*# of constraints*	*Determ. optimal function value*	*Robust optimal function value*	*# of function calls (Determ.)*	*# of function calls (Robust)*	*Solution time† in seconds (Determ.)*	*Solution time in seconds (Robust)*	*Iterations of Robust Benders method*
Example 1 (RMILP)	(5, 3)	12	−23.0	−20.8	5	31	0.351	0.538	3
Example 2 (RMILP)	(5, 3)	12	−31.2	−28.8	5	31	0.271	0.658	3
Example 3 (RMIQP)	(4, 2) x_1, x_2	6	9.2	9.3	7	21	0.356	1.245	4
Example 4 (RMIQP)	(4, 3) x_1, x_2, x_3	6	9.2	10.2	7	21	0.356	1.729	4
Hock 100 (Qconvex)	(8, 3) x_1, x_2, x_3	18	680.6	695.8	8	37	0.688	1.236	4
Hock 106 (Qconvex)	(7, 4) x_1,x_2,x_3,x_4	22	7474.7	7474.8	7	31	0.364	0.681	4
Heat Exch. (Magrab *et al*, 2004)	(9, 1) N_T	89	−1006.7	−964.8	49	1092	1.765	4.657	7
Floudas 1	(2, 3)	5	7.7	8.2	5	11	0.742	0.846	2
Floudas 2	(2, 1)	3	1.1	1.4	5	9	0.301	0.532	2
Floudas 3	(3, 4)	9	4.6	4.9	7	21	0.587	0.836	3
Floudas 4	(3, 8)	7	−0.9	−0.6	14	57	1.287	2.495	6
Floudas 5	(2, 0)	4	31.0	48.0	4	13	0.213	0.258	2
Floudas 6	(1, 1)	3	−17.0	−13.5	5	27	0.247	1.395	2

*For Example 3, Example 4, Hock 100, Hock 106, Heat Exch., certain variables from the source problem are constrained to be integer and are listed under the number of variables for reference.

†All problems were solved on a 2 GHz computer with 4 GB memory using GAMS (GAMS, 2009). For a discussion on function calls, refer to the appendix in Siddiqui (2011).

5. Concluding remarks

This paper presents a new robust optimization approach to solve problems that have mixed-integer decision variables and interval uncertainty. The proposed Robust Benders method obtains optimal robust solutions to MILP, mixed-integer quadratic programming, and a class of extended problems which include any problems that have quasiconvex constraints for which the standard Benders decomposition converges. The approach is computationally tractable and is tested on 13 numerical and engineering examples with the most general being nonlinear (non-convex) objective function and nonlinear (non-convex) constraint mixed-integer robust optimization problems.

This paper provides theoretical and numerical evidence of the method providing promising results. In particular, the theory is a natural extension of standard Benders decomposition that takes advantage of the problem structure. Such an extension helps overcome both the integer aspect as well as the two-level nature of the problem. A general convergence proof for the proposed algorithm is part of ongoing research.

Acknowledgements—The work presented in this paper was supported in part by the Office of Naval Research Contract N000140810384. The work of the third author was also supported in part by the ONR Grant N000141310160. Such support does not constitute an endorsement by the funding agency of the opinions expressed in this paper. We would also like to thank the anonymous referees for their comments.

References

Atamturk A (2006). Strong formulations of robust mixed 0–1 programming. *Mathematical Programming* **108**(2–3): 235–250.

Balling R, Free J and Parkinson A (1986). Consideration of worst-case manufacturing tolerances in design optimization. *Journal of Mechanisms, Transmissions, Automation in Design* **108**(4): 438–441.

Beale E (1965). Survey of integer programming. *Journal of the Operational Research Society* **16**(2): 219–228.

Ben-Tal A and Nemirovski A (2002). Robust optimization—methodology and applications. *Mathematical Programming, Series B* **92**(3): 453–480.

Benders J (1962). Partitioning procedures for solving mixed-variables programming problems. *Numerishe Mathematik* **4**(1): 238–252.

Bertsimas D and Sim M (2004a). The price of robustness. *Operations Research* **52**(1): 35–33.

Bertsimas D and Sim M (2004b). Robust discrete optimization under ellipsoidal uncertainty sets, http://web.mit.edu/dbertsim/www/papers .html, accessed 3 April 2014.

Conejo AJ, Castillo E, Minguez R and Garcia-Bertrand R (2006). *Decomposition Techniques in Mathematical Programming*. Springer: New York, NY.

Floudas CA *et al* (1999). *Handbook of Test Problems in Local and Global Optimization*. Springer: New York, NY.

Gabriel S, Shim Y, Conejo A, de la Torre S and Garcia-Bertrand R (2009). A Benders decomposition method for discretely-constrained mathematical programs with equilibrium constraints. *Journal of the Operational Research Society* **61**(9): 1–16.

GAMS (2009). General Algebraic Modeling System, GAMS Version 22.9, www.gams.com.

Ganzerli S and Pantelides CP (1999). Load and resistance convex models for optimum design. *Structural Optimization* **17**(4): 259–268.

Hock W and Schittkowski K (1980). *Test Examples for Nonlinear Programming Codes*. Springer-Verlag: New York, NY.

Li M, Azarm S and Boyars A (2006). A new deterministic approach using sensitivity region measures for multi-objective robust and feasibility robust design optimization. *Journal of Mechanical Design* **128**(4): 874–883.

Li M, Gabriel S, Shim Y and Azarm S (2011). Interval uncertainty-based robust optimization for convex and non-convex quadratic programs with applications in network infrastructure planning. *Networks and Spatial Economics* **11**(1): 159–191.

Magrab E, Azarm S, Balachandran B, Duncan J, Herold K and Walsh G (2004). *An Engineer's Guide to Matlab*. Prentice-Hall: New York, NY.

Montemanni R (2006). A Benders decomposition approach for the robust spanning tree problem with interval data. *Discrete Optimization* **174**(3): 1479–1490.

Ng TS, Sun. Y and Fowler J (2010). Semiconductor lot allocation using robust optimization. *European Journal of Operational Research* **205**(3): 557–570.

Poojari C, Lucas C and Mitra G (2008). Robust solutions and risk measures for a supply chain planning problem under uncertainty. *Journal of the Operational Research Society* **59**(1): 2–12.

Saito H and Murota K (2007). Benders decomposition approach to robust mixed integer programming. *Pacific Journal of Optimization* **3**(1): 99–112.

Siddiqui S (2011). Solving two-level optimization problems with application to robust design and energy markets. Doctoral Dissertation, University of Maryland, College Park, MD.

Siddiqui S, Azarm S and Gabriel S (2011). A modified Benders decomposition method for efficient robust optimization under interval uncertainty. *Structural and Multidisciplinary Optimization* **44**(2): 259–275.

Soyster AL (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research* **21**(5): 1154–1157.

Wu Y (2011). Modeling of containerized air cargo forwarding problems under uncertainty. *Journal of the Operational Research Society* **62**(7): 1211–1226.

Zeghal F, Haouari M, Sherali H and Aissaoui N (2011). Flexible aircraft fleeting and routing at TunisAir. *Journal of the Operational Research Society* **62**(2): 368–380.

Zhu X and Sherali H (2009). Two-stage workforce planning under demand fluctuations and uncertainty. *Journal of the Operational Research Society* **60**(1): 94–103.

Received 13 July 2012;
accepted 17 March 2014 after two revisions